

SUB-CRITICALITY OF NON-LOCAL SCHRÖDINGER SYSTEMS WITH ANTISYMMETRIC POTENTIALS AND APPLICATIONS TO HALF-HARMONIC MAPS

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Abstract

We consider nonlocal linear Schrödinger-type critical systems of the type

$$\Delta^{1/4}v = \Omega v \quad \text{in } \mathbb{R}. \quad (1)$$

where Ω is antisymmetric potential in $L^2(\mathbb{R}, so(m))$, v is a \mathbb{R}^m valued map and Ωv denotes the matrix multiplication. We show that every solution $v \in L^2(\mathbb{R}, \mathbb{R}^m)$ of (1) is in fact in $L^p_{loc}(\mathbb{R}, \mathbb{R}^m)$, for every $2 \leq p < +\infty$, in other words, we prove that the system (1) which is a-priori only critical in L^2 happens to have a subcritical behavior for antisymmetric potentials. As an application we obtain the $C^{0,\alpha}_{loc}$ regularity of weak 1/2-harmonic maps into C^2 compact manifold without boundary.

Key words. Harmonic maps, nonlinear elliptic PDE's, regularity of solutions, commutator estimates.

MSC 2000. 58E20, 35J20, 35B65, 35J60, 35S99

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1 Introduction

In this paper we consider maps $v = (v_1, \dots, v_m) \in L^2(\mathbb{R}, \mathbb{R}^m)$ solving a system of the form

$$\forall i = 1 \dots m \quad \Delta^{1/4} v^i = \sum_{j=1}^m \Omega_j^i v^j, \quad (2)$$

where $\Omega = (\Omega_i^j)_{i,j=1 \dots m} \in L^2(\mathbb{R}, so(m))$ is an L^2 maps from \mathbb{R} into the space $so(m)$ of $m \times m$ antisymmetric matrices. The operator $\Delta^{1/4}$ on \mathbb{R} is defined by means of the the Fourier transform as follows

$$\widehat{\Delta^{1/4} u} = |\xi|^{1/2} \hat{u},$$

(given a function f , \hat{f} or \mathcal{F} denotes the Fourier transform of f).

We will also simply denote such system in the following way

$$\Delta^{1/4} v = \Omega v.$$

We remark that the system (5) is a-priori critical for $v \in L^2(\mathbb{R})$. Indeed under the assumptions that $v, \Omega \in L^2$ we obtain that $\Delta^{1/4} v \in L^1$ and using classical theory on singular integrals we deduce that $v \in L_{loc}^{2,\infty}$, the weak- L^2 space, which has the same homogeneity of L^2 . Thus we are more or less back to the initial assumption which is a property that characterizes critical equations.

In such critical situation it is a-priori not clear whether solutions have some additional regularity or whether weakly converging sequences of solutions tends to another solution (stability of the equation under weak convergence)...etc.

In [10] and [11] the second author proved the sub-criticality of local *a-priori* critical Schödinger systems of the form

$$\forall i = 1 \dots m \quad -\Delta u^i = \sum_{j=1}^m \Omega_j^i \cdot \nabla u^j, \quad (3)$$

where $u = (u^1, \dots, u^m) \in W^{1,2}(D^2, \mathbb{R}^m)$ and $\Omega \in L^2(D^2, \mathbb{R}^2 \times so(m))$ or of the form

$$\forall i = 1 \dots m \quad -\Delta v^i = \sum_{j=1}^m \Omega_j^i v^j, \quad (4)$$

where $v \in L^{n/(n-2)}(B^n, \mathbb{R}^m)$ and $\Omega \in L^{n/2}(B^n, \mathbb{R}^m)$. In each of these two situations the antisymmetry of Ω was responsible for the regularity of the solutions or for the stability of the system under weak convergence.

Our first main result in this paper is to establish the sub-criticality of non-local Schrödinger systems of the form (2). Precisely we prove the following theorem which extends to a non-local setting the phenomena observed in [10] and [11] for the above local systems.

Theorem 1.1 *Let $\Omega \in L^2(\mathbb{R}, so(m))$ and $v \in L^2(\mathbb{R})$ be a weak solution of*

$$\Delta^{1/4}v = \Omega v. \quad (5)$$

Then $v \in L_{loc}^p(\mathbb{R})$ for every $1 \leq p < +\infty$.

As in the previous works the main technique for proving the above result is to proceed to a *change of gauge* by rewriting the system after having multiplied v by a well chosen rotation valued map $P \in H^{1/2}(\mathbb{R}, SO(m))^{(1)}$ which is "integrating" Ω in an optimal way. In [10] the choice of P for systems of the form (3) was given by the geometrically relevant *Coulomb Gauge* satisfying

$$\operatorname{div} [P^{-1}\nabla P + P^{-1}\Omega P] = 0. \quad (6)$$

Here, like in [11], an appropriate choice of the gauge P satisfies a maybe less geometrically relevant equation which seems however to be better adapted to the system (2) :

$$\operatorname{Asymm} [P^{-1}\Delta^{1/4}P] = 2^{-1} [P^{-1}\Delta^{1/4}P - \Delta^{1/4}P^{-1}P] = \Omega. \quad (7)$$

The local existence of such P is given by the following theorem.

Theorem 1.2 *There exists $\varepsilon > 0$ and $C > 0$ such that for every $\Omega \in L^2(\mathbb{R}; so(m))$ satisfying $\int_{\mathbb{R}} |\Omega|^2 dx \leq \varepsilon$, there exists $P \in \dot{H}^{1/2}(\mathbb{R}, SO(m))$ such that*

$$\begin{aligned} (i) \quad & P^{-1}\Delta^{1/4}P - \Delta^{1/4}P^{-1}P = 2\Omega; \\ (ii) \quad & \int_{\mathbb{R}} |\Delta^{1/4}P|^2 dx \leq C \int_{\mathbb{R}} |\Omega|^2 dx. \end{aligned} \quad (8)$$

□

The proof of this theorem is established following an approach introduced by K.Uhlenbeck in [16] while constructing *Coulomb Gauges* for L^2 curvatures in 4 dimension. The construction does not provide the continuity of the map which to $\Omega \in L^2$ assigns $P \in \dot{H}^{1/2}$. This illustrates the difficulty of the proof of Theorem 1.2 which is not a direct consequence of an application of the local inversion theorem but requires more elaborated arguments.

⁽¹⁾ $SO(m)$ is the space of $m \times m$ matrices R satisfying $R^t R = R R^t = Id$ and $\det(R) = +1$

Thus if the L^2 norm of Ω is small, Theorem 1.2 gives a P for which $w := Pv$ satisfies

$$\begin{aligned}\Delta^{1/4}w &= -[P\Omega P^{-1} - \Delta^{1/4}P P^{-1}] w + N(P, v) \\ &= -\text{symm}(\Delta^{1/4}P P^{-1}) w + N(P, v).\end{aligned}\tag{9}$$

where N is the bilinear operator defined as follows. For an arbitrary integer n , for every $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell \times m}(\mathbb{R}))$ $\ell \geq 0$ ⁽²⁾ and $v \in L^2(\mathbb{R}^n, \mathbb{R}^m)$, N is given by

$$N(Q, v) := \Delta^{1/4}(Qv) - Q\Delta^{1/4}v + \Delta^{1/4}Qv.\tag{10}$$

One of the key result used in [4] establishes that, under the above assumptions on $Q \in H^{1/2}(\mathbb{R}^n, M_m(\mathbb{R}))$ and $v \in L^2(\mathbb{R}^n, \mathbb{R}^m)$, $N(Q, v)$ is more regular than each of it's three generating terms respectively $\Delta^{1/4}(Qv)$, $Q\Delta^{1/4}v$ and $\Delta^{1/4}Qv$ ⁽³⁾. We proved that $N(Q, v)$ is in fact in $H^{-1/2}(\mathbb{R}, \mathbb{R}^m)$. Such a result in [4] was called a 3-commutator estimate (see Theorem 1.3).

In the paper [5] we are improve the gain of regularity by compensation obtained in [4]. In order to make it more precise we recall the definition of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ which is the space of L^1 functions f on \mathbb{R}^n satisfying

$$\int_{\mathbb{R}^n} \sup_{t \in \mathbb{R}} |\phi_t * f|(x) dx < +\infty \quad ,$$

where $\phi_t(x) := t^{-n} \phi(t^{-1}x)$ and where ϕ is some function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x) dx = 1$.⁽⁴⁾

Lemma 1.1 *There exists a constant $C > 0$ such that, for any $Q \in \dot{H}^{1/2}(\mathbb{R}^n, M_m(\mathbb{R}))$ and $v \in L^2(\mathbb{R}^n, \mathbb{R}^m)$, $N(Q, v) = \Delta^{1/4}(Qv) - Q\Delta^{1/4}v + \Delta^{1/4}Qv$ is in $\mathcal{H}^1(\mathbb{R}^n)$ and the following estimate holds*

$$\|N(Q, v)\|_{\mathcal{H}^1} \leq C \|Q\|_{\dot{H}^{1/2}} \|v\|_{L^2(\mathbb{R})}.\tag{11}$$

Thus in equation (9) the last term in the r.h.s happens to be slightly more regular. It remains to deal with the first term in this r.h.s. : $-\text{symm}(\Delta^{1/4}P P^{-1}) w$. A-priori $\text{symm}(\Delta^{1/4}P P^{-1}) = 2^{-1}[\Delta^{1/4}P P^{-1} + P \Delta^{1/4}P^{-1}]$ is only in L^2 but here again we are going to take advantage of a gain of regularity due to a compensation. Though, individually each of the terms $\Delta^{1/4}P P^{-1}$ and it's transposed $P \Delta^{1/4}P^{-1}$ are only in L^2 , the sum happens to belong to the "slightly" smaller space $L^{2,1}$ defined as follows: $L^{2,1}(\mathbb{R})$ is the Lorentz space of measurable functions satisfying

$$\int_{\mathbb{R}_+} t^{-1/2} f^*(t) dt < +\infty \quad ,$$

⁽²⁾ $\mathcal{M}_{\ell \times m}(\mathbb{R})$ denotes, as usual, the space of $\ell \times m$ real matrices.

⁽³⁾ The last one for example being only a-priori in L^1 .

⁽⁴⁾ For more properties on the Hardy space \mathcal{H}^1 we refer to [7] and [8].

where f^* is the decreasing rearrangement of $|f|$.

The fact that $\text{symm}(\Delta^{1/4}P P^{-1})$ belongs to $L^{2,1}(\mathbb{R})$ comes from the combination of the following lemma according to which $\Delta^{1/4}(\text{symm}(\Delta^{1/4}P P^{-1})) \in \mathcal{H}^1(\mathbb{R})$ and the sharp Sobolev embedding ⁽⁵⁾ which says that $f \in \mathcal{H}^1(\mathbb{R})$ implies that $\Delta^{-1/4}f \in L^{2,1}$. Precisely we have

Lemma 1.2 *Let $P \in H^{1/2}(\mathbb{R}, SO(m))$ then $\Delta^{1/4}[\text{symm}(\Delta^{1/4}P P^{-1})]$ is in the Hardy space $\mathcal{H}^1(\mathbb{R})$ and the following estimates hold*

$$\|\Delta^{1/4}[\Delta^{1/4}P P^{-1} + P \Delta^{1/4}P^{-1}]\|_{\mathcal{H}^1} \leq C\|P\|_{H^{1/2}}^2$$

where $C > 0$ is a constant independent of P . This implies in particular that

$$\|\text{symm}(\Delta^{1/4}P P^{-1})\|_{L^{2,1}} \leq C\|P\|_{H^{1/2}}^2.$$

The proof of this lemma is a consequence of the *3-commutator estimates* in [4] (see Theorem 1.5 below).

Remark 1 The fact that, for rotation valued maps $P \in W^{2,n/2}(\mathbb{R}^n, SO(m))$ ($n > 2$), $\text{symm}(\Delta P P^{-1})$ happens to be more regular than $\text{Asymm}(\Delta P P^{-1})$ was also one of the key point in [11].

As we explain in Section 3 Theorem 1.1 is a consequence of this special choice of P for which the new r.h.s. in the gauge transformed equation (9) is slightly more regular due to lemmas 1.1 and lemma 1.2. More precisely this *gain of regularity* in the right of equation (9) combined with suitable localization arguments permit to obtain the following local Morrey type estimate for Pv and thus for v , being P bounded,

$$\sup_{\substack{x_0 \in B(0, \rho) \\ 0 < r < \rho/4}} r^{-\beta} \int_{B(x_0, r)} |\Delta^{1/4}v| dx \leq C \quad (12)$$

for ρ small enough and $0 < \beta < 1/2$ independent on x_0 . Theorem 5.1 in [1] yields that $v \in L_{loc}^q(\mathbb{R})$ for some $q > 2$. ⁽⁶⁾

Our study of the linear systems has been originally motivated by the following non-linear problem.

In the joint paper [4] we proved the $C_{loc}^{0, \alpha}$ regularity of weak 1/2-harmonic maps into a sphere S^{m-1} . The second aim of the present paper is to extend this result to weak 1/2-harmonic maps with values in a k dimensional sub-manifold \mathcal{N} , which is supposed at least

⁽⁵⁾The fact that $v \in \mathcal{H}^1$ implies $\Delta^{-1/4}v \in L^{2,1}$ is deduced by duality from the fact that $\Delta^{1/4}v \in L^{2, \infty}$ implies that $v \in BMO(\mathbb{R})$ - This last embedding has been proved by Adams in [1]

⁽⁶⁾ In a paper in preparation [5] we show that the solutions of (5) are actually in $L_{loc}^\infty(\mathbb{R})$.

C^2 , compact and without boundary. We recall that 1/2-harmonic maps are functions u in the space $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) = \{u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) : u(x) \in \mathcal{N}, \text{ a.e. } \}$, which are critical points for perturbation of the type $\Pi_{\mathcal{N}}^N(u + t\varphi)$, ($\varphi \in C^\infty$ and $\Pi_{\mathcal{N}}^N$ is the normal projection on \mathcal{N}) of the functional

$$\mathcal{L}(u) = \int_{\mathbb{R}} |\Delta^{1/4} u(x)|^2 dx, \quad (13)$$

(see Definition 1.1 in [4]). The Euler Lagrange equation associated to this non linear problem can be written as follows :

$$\Delta^{1/2} u \wedge \nu(u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad (14)$$

where $\nu(z)$ is the Gauss Maps at $z \in \mathcal{N}$ taking values into the grassmannian $\tilde{Gr}_{m-k}(\mathbb{R}^m)$ of oriented $m - k$ planes in \mathbb{R}^m which is given by the oriented normal $m - k$ -plane to $T_z \mathcal{N}$.
(7)

The Euler Lagrange equation in the form (14) is hiding fundamental properties of this equation such as it's elliptic nature...etc and is difficult to use directly for solving problems related to regularity and compactness. One of the first task is then to rewrite it in a form that will make some of it's analysis features more apparent. This is the purpose of the next proposition. Before to state it we need some additional notations

Denote by $P^T(z)$ and $P^N(z)$ the projections respectively to the tangent space $T_z \mathcal{N}$ and to the normal space $N_z \mathcal{N}$ to \mathcal{N} at $z \in \mathcal{N}$. For $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ we simply denote by P^T and P^N the compositions $P^T \circ u$ and $P^N \circ u$. In Section 5 we establish that , under the assumption \mathcal{N} to be C^2 , $P^T \circ u$ as well as $P^N \circ u$ are matrix valued maps in $\dot{H}^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$.

A useful formulation of the 1/2-harmonic map equation is given by the following result

Proposition 1.1 *Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ be a weak 1/2-harmonic map. Then the following equation holds*

$$\Delta^{1/4} v = \tilde{\Omega}_1 + \tilde{\Omega}_2 v + \Omega v, \quad (15)$$

where $v \in L^2(\mathbb{R}, \mathbb{R}^{2m})$ is given by

$$v := \begin{pmatrix} P^T \Delta^{1/4} u \\ \mathcal{R} P^N \Delta^{1/4} u \end{pmatrix}$$

and where \mathcal{R} is the Fourier multiplier of symbol $\sigma(\xi) = i \frac{\xi}{|\xi|}$.

$\Omega \in L^2(\mathbb{R}, so(2m))$ is given by

$$\Omega = 2 \begin{pmatrix} -\omega & \omega_{\mathcal{R}} \\ \omega_{\mathcal{R}} & -\mathcal{R}\omega_{\mathcal{R}} \end{pmatrix}$$

(7) Since we are assuming that \mathcal{N} is C^2 , ν is a C^1 map on \mathcal{N} and the paracomposition gives that $\nu(u)$ is in $\dot{H}^{1/2}(\mathbb{R}, \wedge^{m-k} \mathbb{R}^m)$ hence, since $\Delta^{1/2} u$ is *a-priori* in $\dot{H}^{-1/2}$ the product $\Delta^{1/2} u \wedge \nu(u)$ makes sense in \mathcal{D}' using the duality $\dot{H}^{1/2} - \dot{H}^{-1/2}$

the maps ω and $\omega_{\mathcal{R}}$ are in $L^2(\mathbb{R}, \mathfrak{so}(m))$ and given respectively by

$$\omega = \frac{\Delta^{1/4} P^T P^T - P^T \Delta^{1/4} P^T}{2},$$

and

$$\omega_{\mathcal{R}} = \frac{(\mathcal{R} \Delta^{1/4} P^T) P^T - P^T (\mathcal{R} \Delta^{1/4} P^T)}{2}.$$

Finally the maps $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are respectively in $L^{2,1}(\mathbb{R}, M_{2m}(\mathbb{R}))$ and in $\mathcal{H}^1(\mathbb{R}, \mathbb{R}^{2m})$.

The explicit formulations of $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given in Section 5. The control on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ is a consequence of regularity by compensation results on some operators that we now introduce.

For every $Q, v \in L^2(\mathbb{R}^n)$ we define the operator F by

$$F(Q, v) := \mathcal{R}(Q) \mathcal{R}(v) - Qv. \quad (16)$$

In [3] it is shown that $F(Q, v) \in H^{-1/2}(\mathbb{R})$ and

$$\|F(Q, v)\|_{H^{-1/2}(\mathbb{R})} \leq C \|Q\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}. \quad (17)$$

By a suitable estimate on the dual operator of F (Lemma B.8) we deduce the following sharper estimate

$$\|F(Q, v)\|_{H^{-1/2}(\mathbb{R})} \leq C \|Q\|_{L^2(\mathbb{R})} \|v\|_{L^{2,\infty}(\mathbb{R})}. \quad (18)$$

Next we recall some *commutator estimates* we obtained in [4].

Theorem 1.3 *Let $n \in \mathbb{N}^*$ and let $u \in BMO(\mathbb{R}^n)$, $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell \times m}(\mathbb{R}))$. Denote*

$$T(Q, u) := \Delta^{1/4}(Q \Delta^{1/4} u) - Q \Delta^{1/2} u + \Delta^{1/4} u \Delta^{1/4} Q, \quad ,$$

then $T(Q, u) \in H^{-1/2}(\mathbb{R}^n)$ and there exists $C > 0$, depending only on n , such that

$$\|T(Q, u)\|_{H^{-1/2}(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|u\|_{BMO(\mathbb{R}^n)}. \quad (19)$$

□

Theorem 1.4 *Let $n \in \mathbb{N}^*$ and let $u \in BMO(\mathbb{R}^n)$, $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell \times m}(\mathbb{R}))$. Denote*

$$S(Q, u) := \Delta^{1/4}[Q \Delta^{1/4} u] - \mathcal{R}(Q \nabla u) + \mathcal{R}(\Delta^{1/4} Q \mathcal{R} \Delta^{1/4} u)$$

where \mathcal{R} is the Fourier multiplier of symbol $m(\xi) = i \frac{\xi}{|\xi|}$. Then $S(Q, u) \in H^{-1/2}(\mathbb{R}^n)$ and there exists C depending only on n such that

$$\|S(Q, u)\|_{H^{-1/2}(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|u\|_{BMO(\mathbb{R}^n)}. \quad (20)$$

□

As it is observed in [4] Theorems 1.3 and 1.4 are consequences respectively of the following results which are their “dual versions” .

Theorem 1.5 *Let $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$, denote*

$$R(Q, u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u) .$$

then $R(Q, u) \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|R(Q, u)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)} . \quad (21)$$

Theorem 1.6 *Let $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$, denote*

$$\tilde{S}(Q, u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \nabla(Q\mathcal{R}u) + \mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u) .$$

Then $\tilde{S}(Q, u) \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|\tilde{S}(Q, u)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)} . \quad (22)$$

□

Since the operators R and \tilde{S} are the duals respectively of T and S , by combining Theorems 1.3 and 1.5 and Theorems 1.4 and 1.6 one gets the followings sharper estimates for T and S :

$$\|T(Q, u)\|_{H^{-1/2}(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|\Delta^{1/4}u\|_{L^{2,\infty}(\mathbb{R}^n)} ; \quad (23)$$

$$\|S(Q, u)\|_{H^{-1/2}(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|\Delta^{1/4}u\|_{L^{2,\infty}(\mathbb{R}^n)} . \quad (24)$$

An adaptation of theorem 1.1 to the Euler Lagrange equation of the 1/2-Energy written in the form (15) leads to the following theorem which is the second main result of the present paper.

Theorem 1.7 *Let \mathcal{N} be a closed C^2 submanifold of \mathbb{R}^m . Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ be a weak 1/2-harmonic map into \mathcal{N} , then $u \in C_{loc}^{0,\alpha}(\mathbb{R}, \mathcal{N})$. □*

Finally a classical elliptic type bootstrap argument leads to the following result (see [5] for the details of this argument).

Theorem 1.8 *Let \mathcal{N} be a smooth closed submanifold of \mathbb{R}^m . Let u be a weak 1/2-harmonic map in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$, then u is C^∞ . □*

The regularity of critical points of non-local functionals has been recently investigated by Moser [9]. In this work critical points to the functional that assigns to any $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ the minimal Dirichlet energy among all possible extensions in \mathcal{N} are considered, while in the present paper the classical $\dot{H}^{1/2}$ Lagrangian corresponds to the minimal Dirichlet energy among all possible extensions in \mathbb{R}^m . Hence the approach in [9] consists in working with

an intrinsic version of $H^{1/2}$ –energy while we are considering here an extrinsic one. The drawback of considering the intrinsic energy is that the Euler Lagrange equation is almost impossible to write explicitly and is then implicit while in the present case it has the explicit form (14). However the intrinsic version of the $1/2$ –harmonic map is more closely related to the existing regularity theory of Dirichlet Energy minimizing maps into \mathcal{N} .

The paper is organized as follows.

- In Section 3 we prove Theorem 1.1 .
- In Section 4 we prove Theorem 1.2 .
- In Section 5 we derive the Euler-Lagrange equation (15) associated to the Lagrangian (13) and we prove Theorem 1.7 .
- In Appendix A we prove some L –energy decrease control for solutions to linear non-local Schrödinger type systems .
- In Appendix B we provide commutator estimates that are crucial for the construction of the gauge P .

2 Preliminaries: function spaces and the fractional Laplacian

In this Section we introduce some notations and definitions we are going to use in the sequel.

For $n \geq 1$, we denote respectively by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the spaces of Schwartz functions and tempered distributions. Moreover given a function v we will denote either by \hat{v} or by $\mathcal{F}[v]$ the Fourier Transform of v :

$$\hat{v}(\xi) = \mathcal{F}[v](\xi) = \int_{\mathbb{R}^n} v(x) e^{-i\langle \xi, x \rangle} dx .$$

Throughout the paper we use the convention that x, y denote variables in the space and ξ the variable in the phase .

We recall the definition of fractional Sobolev space (see for instance [15]).

Definition 2.1 *For a real $s \geq 0$,*

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : |\xi|^s \mathcal{F}[v] \in L^2(\mathbb{R}^n)\}$$

For a real $s < 0$,

$$H^s(\mathbb{R}^n) = \{v \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^s \mathcal{F}[v] \in L^2(\mathbb{R}^n)\} .$$

□

It is known that $H^{-s}(\mathbb{R}^n)$ is the dual of $H^s(\mathbb{R}^n)$.

For $0 < s < 1$ another classical characterization of $H^s(\mathbb{R}^n)$ which does not make use of the Fourier transform is the following, (see for instance [15]).

Lemma 2.1 *For $0 < s < 1$, $u \in H^s(\mathbb{R}^n)$ is equivalent to $u \in L^2(\mathbb{R}^n)$ and*

$$\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \right) dx dy \right)^{1/2} < +\infty.$$

□

For $s > 0$ we set

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + |\xi|^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^n)},$$

and

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} = |\xi|^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^n)}.$$

For an open set $\Omega \subset \mathbb{R}^n$, $H^s(\Omega)$ is the space of the restrictions of functions from $H^s(\mathbb{R}^n)$ and

$$\|u\|_{\dot{H}^s(\Omega)} = \inf \left\{ \|U\|_{\dot{H}^s(\mathbb{R}^n)}, \quad U = u \text{ on } \Omega \right\}$$

In the case $0 < s < 1$ then $u \in H^s(\Omega)$ if and only if $u \in L^2(\Omega)$ and

$$\left(\int_{\Omega} \int_{\Omega} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \right) dx dy \right)^{1/2} < +\infty.$$

Moreover

$$\|u\|_{\dot{H}^s(\Omega)} \simeq \left(\int_{\Omega} \int_{\Omega} \left(\frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \right) dx dy \right)^{1/2} < +\infty,$$

see for instance [15].

Finally for a submanifold \mathcal{N} of \mathbb{R}^m we can define

$$H^s(\mathbb{R}^n, \mathcal{N}) = \{u \in H^s(\mathbb{R}^n, \mathbb{R}^m) : u(x) \in \mathcal{N}, \text{ a.e.}\}.$$

Given $q > 2$ we also set

$$W^{s,q}(\mathbb{R}^n) := \{v \in L^q(\mathbb{R}^n) : |\xi|^s \mathcal{F}[v] \in L^q(\mathbb{R}^n)\}.$$

We shall make use of the Littlewood-Paley dyadic decomposition of unity that we recall here. Such a decomposition can be obtained as follows. Let $\phi(\xi)$ be a radial Schwartz function supported in $\{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$, which is equal to 1 in $\{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$. Let $\psi(\xi)$ be the function given by

$$\psi(\xi) := \phi(\xi) - \phi(2\xi).$$

ψ is then a "bump function" supported in the annulus $\{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$.

Let $\psi_0 = \phi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions ψ_j , for $j \in \mathbb{Z}$, are supported in $\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and they realize a dyadic decomposition of the unity :

$$\sum_{j \in \mathbb{Z}} \psi_j(x) = 1.$$

We further denote

$$\phi_j(\xi) := \sum_{k=-\infty}^j \psi_k(\xi).$$

The function ϕ_j is supported on $\{\xi, |\xi| \leq 2^{j+1}\}$.

We recall the definition of the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and homogeneous Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$ in terms of the above dyadic decomposition.

Definition 2.2 Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we set

$$\begin{aligned} \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} &= \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} & \text{if } q < \infty \\ \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} &= \sup_{j \in \mathbb{Z}} 2^{js} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty \end{aligned} \quad (25)$$

When $p, q < \infty$ we also set

$$\|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]|^q \right)^{1/q} \right\|_{L^p}.$$

□

The space of all tempered distributions u for which the quantity $\|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}$ is finite is called the homogeneous Besov space with indices s, p, q and it is denoted by $\dot{B}_{p,q}^s(\mathbb{R}^n)$. The space of all tempered distributions f for which the quantity $\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)}$ is finite is called the homogeneous Triebel-Lizorkin space with indices s, p, q and it is denoted by $\dot{F}_{p,q}^s(\mathbb{R}^n)$. A classical result says ⁽⁸⁾ that $\dot{W}^{s,q}(\mathbb{R}^n) = \dot{B}_{q,2}^s(\mathbb{R}^n) = \dot{F}_{q,2}^s(\mathbb{R}^n)$.

Finally we denote $\mathcal{H}^1(\mathbb{R}^n)$ the homogeneous Hardy Space in \mathbb{R}^n . A less classical results ⁽⁹⁾ asserts that $\mathcal{H}^1(\mathbb{R}^n) \simeq \dot{F}_{2,1}^0$ thus we have

$$\|u\|_{\mathcal{H}^1(\mathbb{R}^n)} \simeq \int_{\mathbb{R}} \left(\sum_j |\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]|^2 \right)^{1/2} dx.$$

⁽⁸⁾ See for instance [7]

⁽⁹⁾ See for instance [8].

We recall that in dimension $n = 1$, the space $\dot{H}^{1/2}(\mathbb{R})$ is continuously embedded in the Besov space $\dot{B}_{\infty,\infty}^0(\mathbb{R})$. More precisely we have

$$\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R}) \hookrightarrow \dot{B}_{\infty,\infty}^0(\mathbb{R}), \quad (26)$$

where $BMO(\mathbb{R})$ is the space of bounded mean oscillation dual to $\mathcal{H}^1(\mathbb{R}^n)$ (see for instance [14], page 31).

The s -fractional Laplacian of a function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as a pseudo differential operator of symbol $|\xi|^{2s}$:

$$\widehat{\Delta^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi). \quad (27)$$

In the case where $s = 1/2$, we can write $\Delta^{1/2}u = -\mathcal{R}(\nabla u)$ where \mathcal{R} is Fourier multiplier of symbol $\frac{i}{|\xi|} \sum_{k=1}^n \xi_k$:

$$\widehat{\mathcal{R}X}(\xi) = \frac{1}{|\xi|} \sum_{k=1}^n i\xi_k \hat{X}_k(\xi)$$

for every $X: \mathbb{R}^n \rightarrow \mathbb{R}^n$, namely $\mathcal{R} = \Delta^{-1/2} \operatorname{div}$.

We denote by $B_r(\bar{x})$ the ball of radius r and centered at \bar{x} . If $\bar{x} = 0$ we simply write B_r . If $x, y \in \mathbb{R}^n$, $x \cdot y$ denote the scalar product between x, y .

For every function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ we denote by $M(u)$ the maximal function of u , namely

$$M(u) = \sup_{r>0, x \in \mathbb{R}^n} |B(x, r)|^{-1} \int_{B(x, r)} |u(y)| dy. \quad (28)$$

3 Regularity of nonlocal Schrödinger type systems

In this Section we prove Theorem 1.1. The proof is based in particular on the *localization estimates* established in Appendix A as well on the *3 commutator estimates* (23) and (21).

Proof of theorem 1.1.

Let $\rho > 0$ be such that $\|\mathbb{1}_{B(0,\rho)}\Omega\|_{L^2} \leq \varepsilon_0$, with ε_0 small enough. We decompose Ω as follows $\Omega_1 = \mathbb{1}_{B(0,\rho)}\Omega$ and $\Omega_2 = (1 - \mathbb{1}_{B(0,\rho)})\Omega$.

Let $P \in \dot{H}^{1/2}(\mathbb{R}, so(m))$ given by Theorem 1.2. We have

$$\Delta^{1/4}(Pv) = [P\Omega_1 P^{-1} - \Delta^{1/4} P P^{-1}] Pv + N(P, v) \quad (29)$$

where N is the operator defined in lemma 10.

Since P satisfies (8)(i) we have

$$\begin{aligned} P\Omega P^{-1} - \Delta^{1/4} P P^{-1} &= -\frac{\Delta^{1/4} P P^{-1} + P \Delta^{1/4} P^{-1}}{2} \\ &= -\operatorname{symm}(\Delta^{1/4} P P^{-1}). \end{aligned} \quad (30)$$

From Theorem 1.5 it follows that

$$\Delta^{1/4}[\text{symm}(\Delta^{1/4}PP^{-1})] = \Delta^{1/4}[(\Delta^{1/4}P)P^{-1}] + \Delta^{1/4}[P\Delta^{1/4}P^{-1}] - \Delta^{1/2}(PP^{-1}) \in \mathcal{H}^1(\mathbb{R}).$$

since $PP^{-1} = Id$. Thus $\text{symm}(\Delta^{1/4}PP^{-1}) \in L^{2,1}(\mathbb{R})$.⁽¹⁰⁾

Claim 1. From Theorems 1.3 and 1.5 we can deduce the estimate (23), which can be expressed in term of the operator N as follows:

$$\|N(Q, v)\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \leq C\|v\|_{L^{2,\infty}(\mathbb{R}^n)} \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}.$$

for every $Q \in \dot{H}^{1/2}(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^n)$.

Proof of Claim 1.

$$\begin{aligned} \|N(Q, v)\|_{\dot{H}^{-1/2}(\mathbb{R}^n)} &= \sup_{\|h\|_{\dot{H}^{1/2}} \leq 1} \int_{\mathbb{R}^n} N(Q, v) h dx \\ &= \sup_{\|h\|_{\dot{H}^{1/2}} \leq 1} \int_{\mathbb{R}^n} v [Q(\Delta^{1/4}h) - \Delta^{1/4}(Qh) + (\Delta^{1/4}Q)h] dx \\ &= \sup_{\|h\|_{\dot{H}^{1/2}} \leq 1} \int_{\mathbb{R}^n} v \Delta^{-1/4}(R(Q, h)) dx \end{aligned}$$

And using Theorem 1.5 we obtain

$$\begin{aligned} \|N(Q, v)\|_{\dot{H}^{-1/2}(\mathbb{R}^n)} &\lesssim \|h\|_{\dot{H}^{1/2}} \|v\|_{L^{2,\infty}} \|Q\|_{\dot{H}^{1/2}} \\ &\lesssim \|v\|_{L^{2,\infty}} \|Q\|_{\dot{H}^{1/2}}. \end{aligned}$$

which concludes the proof of claim 1. □

We set now $w = Pv$ and $\omega = -\text{symm}(\Delta^{1/4}PP^{-1})$ and rewrite equation (29) as follows

$$\Delta^{1/4}w = \omega w + N(P, P^{-1}w) + \Omega_2 P^{-1}w. \quad (31)$$

where by construction $\|\omega\|_{L^{2,1}}, \|P\|_{\dot{H}^{1/2}} \leq \varepsilon_0$.

Claim 2 : *There exists $q > 2$ such that $v \in L_{loc}^q(\mathbb{R})$.*

In order to establish the claim 2, a "natural approach" would be to try to prove directly a *Morrey decrease* for the L^2 norm of v (or equivalently the L^2 norm of w), that is an estimate of the form

$$\sup_{x_0 \in B(0, \rho/4), r > 0} r^{-\beta} \left[\int_{B(x_0, r)} |v|^2 dx \right]^{1/2} < +\infty.$$

⁽¹⁰⁾We recall that $v \in \mathcal{H}^1$ implies $\Delta^{-1/4}v \in L^{2,1}$ see a footnote in the introduction.

for some positive constant $\beta > 0$. We however failed to work directly with the L^2 norm. We have been instead more successful in working with it's weak version : the $L^{2,\infty}$ -norm. Precisely we are going to establish the following bound

$$\sup_{x_0 \in B(0, \rho/4), r > 0} r^{-\beta} \|w\|_{L^{2,\infty}(B(x_0, r))} < +\infty.$$

Let $x_0 \in B(0, \rho/4)$ and $r \in (0, \rho/8)$. We argue by duality and multiply (31) by ϕ which is given as follows. Let $g \in L^{2,1}(\mathbb{R})$, with $\|g\|_{L^{2,1}} \leq 1$ and set $g_{r\alpha} = \mathbb{1}_{B(x_0, r\alpha)}g$, with $0 < \alpha < 1/4$ and $\phi = \Delta^{-1/4}(g_{r\alpha}) \in L^\infty(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})$. We take the scalar product of both sides of equation (31) with ϕ and we integrate.

Left hand side of the equation (31):

$$\begin{aligned} \sup_{\|g\|_{L^{2,1}} \leq 1} \int_{\mathbb{R}} \phi \Delta^{1/4} w dx &= \sup_{\|g\|_{L^{2,1}} \leq 1} \int_{\mathbb{R}} g_{r\alpha} w dx \\ &= \|w\|_{L^{2,\infty}(B(x_0, r\alpha))}. \end{aligned} \quad (32)$$

Right hand side of the equation (31):

We apply Lemmae A.1 , A.3 and A.4 and we respectively obtain in one hand

$$\begin{aligned} \int_{\mathbb{R}} \phi \omega w dx &\leq \|\omega\|_{L^{2,1}} \|g\|_{L^{2,1}} \|w\|_{L^{2,\infty}(B(x_0, r))} \\ &\quad + \sum_{h=-1}^{+\infty} 2^{-h/2} \alpha^{1/2} \|\omega\|_{L^{2,1}} \|g\|_{L^{2,1}} \|w\|_{L^{2,\infty}(B(x_0, 2^{h+1}r) \setminus B(x_0, 2^{h-1}r))} \\ &\lesssim \varepsilon_0 \|w\|_{L^{2,\infty}(B(x_0, r))} + \alpha^{1/2} \sum_{h=-1}^{+\infty} 2^{-h/2} \|w\|_{L^{2,\infty}(B(x_0, 2^{h+1}r) \setminus B(x_0, 2^{h-1}r))}. \end{aligned} \quad (33)$$

In the other hand

$$\begin{aligned} \int_{\mathbb{R}} \phi N(P, P^{-1}w) dx &\leq \varepsilon_0 \|w\|_{L^{2,\infty}(B(x_0, r))} \\ &\quad + C \alpha^{1/2} \sum_{h=1}^{+\infty} 2^{-h/2} \|w\|_{L^{2,\infty}(B(x_0, 2^{h+1}r) \setminus B(x_0, 2^{h-1}r))}, \end{aligned} \quad (34)$$

and finally

$$\int_{\mathbb{R}} \Omega_2 P^{-1} w \phi dx \leq C \alpha^{1/2} r^{1/2}. \quad (35)$$

Thus combining (32)...(34) we get

$$\begin{aligned} \|w\|_{L^{2,\infty}(B(x_0, r\alpha))} &\lesssim \varepsilon_0 \|w\|_{L^{2,\infty}(B(x_0, r))} \\ &\quad + \alpha^{1/2} \sum_{h=1}^{+\infty} 2^{-h/2} \|w\|_{L^{2,\infty}(B_{2^{h+1}r}(x_0) \setminus B_{2^{h-1}r}(x_0))} + \alpha^{1/2} r^{1/2}. \end{aligned} \quad (36)$$

If α and ε are small enough the formula (36) implies that for all $x_0 \in B(0, \rho/4)$ and $0 < r < \rho/8$ we have $\|w\|_{L^{2,\infty}(B(x_0,r))} \leq Cr^\beta$, for some $\beta \in (0, 1/2)$. Since $P \in L^\infty$, this implies that

$$\sup_{\substack{x_0 \in B(0, \rho/4) \\ r > 0}} r^{-\beta} \int_{B(x_0, r)} |\Delta^{1/4} v| dx < +\infty. \quad (37)$$

Theorem 5.1 in [1] yields that $v \in L_{loc}^q(\mathbb{R})$ for some $q > 2$ which finishes the proof of claim 2.

Claim 3: $v \in L_{loc}^p(\mathbb{R})$ for every $p > 2$.

To this end we consider again $\rho > 0$ such that $\|\mathbb{1}_{B(0,\rho)}\Omega\|_{L^2} \leq \varepsilon_0$, with ε_0 small enough. We decompose Ω as follows. Let $\Omega_1 = \mathbb{1}_{B(0,\rho)}\Omega$ and $\Omega_2 = (1 - \mathbb{1}_{B(0,\rho)})\Omega$. We consider an arbitrary $q > 2$ such that $v \in L_{loc}^q$.

Let $x_0 \in B(0, \rho/4)$, $r \in (0, \rho/8)$, $g \in L^{\frac{q}{q-1}}(\mathbb{R})$, with $\|g\|_{L^{\frac{q}{q-1}}} \leq 1$ and set $g_{r\alpha} = \mathbb{1}_{B(x_0, r\alpha)}g$, with $0 < \alpha < 1/4$ and $\phi = \Delta^{-1/4}(g_{r\alpha})$. We write the equation (5) as follows

$$\begin{aligned} \Delta^{1/4} v &= \Omega_1 \mathbb{1}_{B(x_0, r/2)} v + \sum_{h=0}^{+\infty} \Omega_1 \mathbb{1}_{B(x_0, 2^{h+1}r) \setminus B(x_0, 2^h r)} v \\ &+ \Omega_2 v. \end{aligned} \quad (38)$$

We take the scalar product of the equation (38) with $\Delta^{-1/4}(g_{r\alpha})$ and integrate. By arguing as above, one gets

$$\sup_{\substack{x_0 \in B(0, \rho/4) \\ r > 0}} r^{-\gamma} \left[\int_{B(x_0, r)} |v|^q dx \right]^{1/q} < +\infty \quad (39)$$

with $0 < \gamma < 1/4$ independent on q . Thus by injecting (39) in the equation (5) we obtain for the same $\gamma > 0$ independent of q

$$\sup_{\substack{x_0 \in B(0, \rho/4) \\ r > 0}} r^{-\gamma} \|\Delta^{1/4} v\|_{L^{2q/(q+2)}(B(x_0, r))} dx < +\infty. \quad (40)$$

Theorem 3.1 in [1] yields that $v \in L_{loc}^{\tilde{q}}$, with $\tilde{q} > q$. is given by

$$\tilde{q}^{-1} = q^{-1} - 2^{-1}[\gamma^{-1}(q^{-1} + 2^{-1}) - 1]^{-1}.$$

Since $q > 2$ we have

$$\tilde{q}^{-1} < q^{-1} - \frac{2\gamma}{1 - 4\gamma}.$$

By repeating the above arguments with q replaced by \tilde{q} one finally gets that $v \in L_{loc}^p$ for every $p > 2$. This concludes the proof of theorem 1.1. \square

4 Construction of an optimal gauge P : the proof of theorem 1.2.

Proof of Theorem 1.2.

We follow the strategy of [11] in order to construct solutions to $Asymm(P^{-1} \Delta P) = \Omega$ which was itself inspired by Uhlenbeck's construction of Coulomb Gauges solving (6).

Let $2 < q < +\infty$ and denote $1 < q' < 2$ to be the conjugate of q : $(q)^{-1} + (q')^{-1} = 1$. We consider

$$\mathcal{U}_\varepsilon^q = \left\{ \Omega \in L^q(\mathbb{R}, so(m)) \cap L^{q'}(\mathbb{R}, so(m)) : \int_{\mathbb{R}} |\Omega|^2 dx \leq \varepsilon \right\}.$$

Claim: *There exist $\varepsilon > 0$ small enough and $C > 0$ large enough such that*

$$\mathcal{V}_{\varepsilon, C}^q := \left\{ \begin{array}{l} \Omega \in \mathcal{U}_\varepsilon^q : \text{there exists } P \text{ satisfying (8) (i)-(ii)} \\ \text{and} \quad \int_{\mathbb{R}} |\Delta^{1/4} P|^q dx \leq C \int_{\mathbb{R}} |\Omega|^q dx \end{array} \right\}$$

is open and closed in $\mathcal{U}_\varepsilon^q$ and thus $\mathcal{V}_\varepsilon^q \equiv \mathcal{U}_\varepsilon^q$ ($\mathcal{U}_\varepsilon^q$ being path connected).

Proof of the claim

We first observe that $\mathcal{V}_{\varepsilon, C}^q \neq \emptyset$, ($0 \in \mathcal{V}_{\varepsilon, C}^q$).

Step 1: *For any $\varepsilon > 0$ and $C > 0$, $\mathcal{V}_{\varepsilon, C}^q$ is closed in $L^q \cap L^{q'}(\mathbb{R}, so(m))$.*

Let $\Omega_n \in \mathcal{V}_{\varepsilon, C}^q$ such that $\Omega_n \rightarrow \Omega_\infty$ in the norm $L^q \cap L^{q'}$, as $n \rightarrow +\infty$ and let P_n be a solution of

$$P_n^{-1} \Delta^{1/4} P_n - \Delta^{1/4} P_n^{-1} P_n = 2\Omega_n$$

$$\int_{\mathbb{R}} |\Delta^{1/4} P_n|^2 dx \leq C_0 \int_{\mathbb{R}} |\Omega_n|^2 dx,$$

Since $\Omega_n \rightarrow \Omega_\infty$ in the norm $L^q \cap L^{q'}$ and $\int_{\mathbb{R}} |\Omega_n|^2 dx \leq \varepsilon$, we can pass to the limit in this inequality and we have

$$\int_{\mathbb{R}} |\Omega_\infty|^2 dx \leq \varepsilon \tag{41}$$

which implies that $\Omega_\infty \in \mathcal{U}_\varepsilon$.

One can extract a subsequence $P_{n'} \rightharpoonup P_\infty$ in $\dot{H}^{1/2}$. By Rellich-Kondrachov Theorem we also have $P_{n'} \rightarrow P_\infty$ in L_{loc}^2 and hence $P_\infty \in SO(m)$ a.e. Thus $P_\infty \in \dot{H}^{1/2}(\mathbb{R}, SO(m))$ and the lower semi-continuity of the $\dot{H}^{1/2}$ and $\dot{W}^{1/2, q}$ norms implies that

$$\begin{aligned} \int_{\mathbb{R}} |\Delta^{1/4} P_\infty|^2 dx &\leq C_0 \int_{\mathbb{R}} |\Omega_\infty|^2 dx \\ \text{and} \quad \int_{\mathbb{R}} |\Delta^{1/4} P_\infty|^q dx &\leq C_0 \int_{\mathbb{R}} |\Omega_\infty|^q dx. \end{aligned} \tag{42}$$

We have

$$P_n^{-1}\Delta^{1/4}P_n - \Delta^{1/4}P_n^{-1}P_n \rightarrow P_\infty^{-1}\Delta^{1/4}P_\infty - \Delta^{1/4}P_\infty^{-1}P_\infty \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Since $P_n^{-1}\Delta^{1/4}P_n - \Delta^{1/4}P_n^{-1}P_n = \Omega_n \rightarrow \Omega_\infty$ in \mathcal{D}' as well, we deduce that

$$P_\infty^{-1}\Delta^{1/4}P_\infty - \Delta^{1/4}P_\infty^{-1}P_\infty = \Omega_\infty \quad \text{a.e.} \quad (43)$$

and combining (41), (42) and (43) we deduce that $\Omega_\infty \in \mathcal{V}_{\varepsilon, C}^q$ which concludes the proof of Step 1.

Step 2: For $\varepsilon > 0$ small enough and $C > 0$ large enough $\mathcal{V}_{\varepsilon, C}^q$ is open.

For every $P_0 \in \dot{W}^{1/2, q}(\mathbb{R}, SO(m)) \cap \dot{H}^{1/2}(\mathbb{R}, SO(m))$ we introduce the map

$$F^{P_0} : \dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}(\mathbb{R}, so(m)) \longrightarrow L^q \cap L^{q'}(\mathbb{R}, so(m))$$

$$U \longrightarrow (P_0 \exp U)^{-1} \Delta^{1/4} (P_0 \exp U) - \Delta^{1/4} (P_0 \exp U)^{-1} (P_0 \exp U).$$

We claim first that F^{P_0} is a C^1 map between the two Banach spaces $\dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}(\mathbb{R}, so(m))$ and $L^q \cap L^{q'}(\mathbb{R}, so(m))$

- i) Since $\dot{W}^{1/2, q}$ for $q > 2$ embeds continuously in C^0 , the map $V \rightarrow \exp(V)$ is clearly smooth from $\dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}(\mathbb{R}, so(m))$ into $\dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}(\mathbb{R}, SO(m))$.
- ii) The operator $\Delta^{1/4}$ is a smooth linear map from $\dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}(\mathbb{R}, M_m(\mathbb{R}))$ into $L^q \cap L^{q'}(\mathbb{R}, M_m(\mathbb{R}))$.
- iii) Since again $\dot{W}^{1/2, q}$ embeds continuously in L^∞ - $\dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}$ is an algebra - the following map

$$\begin{aligned} \Pi : \dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}(\mathbb{R}, M_n(\mathbb{R})) \times L^q \cap L^{q'}(\mathbb{R}, M_n(\mathbb{R})) &\longrightarrow L^q \cap L^{q'}(\mathbb{R}, M_n(\mathbb{R})) \\ (A, B) &\longrightarrow AB \end{aligned}$$

is also smooth.

Now we show that $dF_0^{P_0} = L^{P_0}$ ⁽¹¹⁾

$$\begin{aligned} L^{P_0}(\eta) &:= -\eta P_0^{-1} \Delta^{1/4} P_0 + \Delta^{1/4} (\eta P_0^{-1}) P_0 \\ &\quad + P_0^{-1} \Delta^{1/4} (P_0 \eta) - \Delta^{1/4} P_0^{-1} P_0 \eta. \end{aligned}$$

⁽¹¹⁾In order to define L^{P_0} as a map from $\dot{W}^{1/2, q} \cap \dot{W}^{1/2, q'}$ into $L^q \cap L^{q'}$ we recall again that we make use of the embedding $\dot{W}^{1/2, q}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ if $q > 2$ (see for instance [14], pag 33).

• **Differentiability of F^{P_0} at $U = 0$:**

$$\begin{aligned} \|F^{P_0}(\eta) - F^{P_0}(0) - L^{P_0} \cdot \eta\|_{L^q \cap L^{q'}} &= \|F^{P_0}(\eta) - F^{P_0}(0) + \eta P_0^{-1} \Delta^{1/4} P_0 \\ &\quad - \Delta^{1/4}(\eta P_0^{-1}) P_0 - P_0^{-1} \Delta^{1/4}(P_0 \eta) + \Delta^{1/4} P_0^{-1} P_0 \eta\|_{L^q \cap L^{q'}} \end{aligned}$$

First of all we estimate

$$\begin{aligned} &\|(P_0 \exp(\eta))^{-1} \Delta^{1/4}(P_0 \exp U \eta) - P_0^{-1} \Delta^{1/4} P_0 + \eta P_0^{-1} \Delta^{1/4} P_0 - P_0^{-1} \Delta^{1/4}(\eta P_0)\|_{L^q \cap L^{q'}} \\ &\leq \|\Delta^{1/4}(P_0)\|_{L^q \cap L^2} \|(P_0 \exp(\eta))^{-1} - P_0^{-1} + \eta(P_0)^{-1}\|_{L^\infty} \\ &\quad + \|(P_0 \exp(\eta))^{-1}\|_{L^\infty} \|\Delta^{1/4}(P_0 \exp(\eta)) - \Delta^{1/4}(P_0) - \Delta^{1/4}(P_0 \eta)\|_{L^q \cap L^{q'}} \\ &\quad + \|\Delta^{1/4}(P_0 \eta)\|_{L^q \cap L^{q'}} \|P_0 \exp(\eta) - P_0\|_{L^\infty} \\ &\leq C o(\|\eta\|_{\dot{W}^{1/2,q}(\mathbb{R})}) \end{aligned} \tag{44}$$

The estimate of

$$\|(P_0 \exp \eta)^{-1} \Delta^{1/4}(P_0 \exp(\eta)) - P_0^{-1} \Delta^{1/4}(P_0) - P_0^{-1} \Delta^{1/4}(P_0 \eta) + \Delta^{1/4} P_0^{-1} P_0 \eta\|_{L^q \cap L^{q'}} .$$

is analogous. Hence we have proved that $dF_0^{P_0} := L^{P_0}$.

• **$d_0 F^{P_0}$ is an isomorphism from $\dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}(\mathbb{R}, so(m))$ into $L^q \cap L^{q'}(\mathbb{R}, so(m))$:**

Precisely we prove the following lemma.

Lemma 4.1 *There exists $\varepsilon > 0$ such that if $\Omega_0 \in \mathcal{V}_{\varepsilon,C}^q$ and P_0 is solution of (8)(i), then for every $\omega \in L^q \cap L^{q'}(\mathbb{R}, so(m))$ there exists a unique $\eta \in \dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}(\mathbb{R}, so(m))$ such that*

$$\omega = -\eta P_0^{-1} \Delta^{1/4} P_0 + \Delta^{1/4}(\eta P_0^{-1}) P_0 + P_0^{-1} \Delta^{1/4}(P_0 \eta) - \Delta^{1/4} P_0^{-1} P_0 \eta \tag{45}$$

and

$$\|\eta\|_{\dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}} \leq C \|\omega\|_{L^q \cap L^{q'}} .$$

□

Proof of Lemma 4.1. We first observe that since $\Omega_0 \in \mathcal{V}_{\varepsilon,C}^q$, then

$$\int_{\mathbb{R}} |\Delta^{1/4} P_0|^2 dx \leq C \int_{\mathbb{R}} |\Omega_0|^2 dx \leq C \varepsilon \tag{46}$$

$$\text{and} \quad \int_{\mathbb{R}} |\Delta^{1/4} P_0|^q dx \leq C \int_{\mathbb{R}} |\Omega_0|^q dx . \tag{47}$$

Claim 1. Let $1 < r < 2$. L^{P_0} is an isomorphism between $\dot{W}^{1/2,r}(\mathbb{R}, so(m))$ and L^r , namely for any $\omega \in L^r(\mathbb{R}, so(m))$ there exists a unique $\eta \in \dot{W}^{1/2,r}(\mathbb{R}, so(m))$ solution to $L^{P_0}(\eta) = \omega$ and

$$\|\eta\|_{\dot{W}^{1/2,r}} \leq C \|\omega\|_{L^r}$$

for $C > 0$.

We rewrite the equation (45) in the following way

$$\begin{aligned} \omega &= 2\Delta^{1/4}\eta - 2\eta P_0^{-1}\Delta^{1/4}P_0 - 2\Delta^{1/4}P_0^{-1}P_0\eta \\ &\quad + Q(\eta, P_0) - Q^t(\eta, P_0), \end{aligned} \quad (48)$$

where

$$Q(\eta, P_0) = \Delta^{1/4}(\eta P_0^{-1})P_0 + \eta P_0^{-1}\Delta^{1/4}P_0 - \Delta^{1/4}\eta. \quad (49)$$

From Lemma B.4 and Lemma B.5 it follows that

$$\|Q(\eta, P_0)\|_{L^r} \leq C \|\eta\|_{\dot{W}^{1/2,r}} \left(\|P_0\|_{\dot{H}^{1/2}} + \|P_0\|_{\dot{H}^{1/2}}^2 \|P_0\|_{L^\infty} \right). \quad (50)$$

Since $2^{-1} + (2-r)(2r)^{-1} = r^{-1}$, by applying Hölder Inequality we get

$$\|\eta P_0^{-1}\Delta^{1/4}P_0\|_{L^r} \leq \|\eta\|_{L^{2r/(2-r)}} \|P_0^{-1}\Delta^{1/4}P_0\|_{L^2}. \quad (51)$$

Thus, since $\dot{W}^{1/2,r}(\mathbb{R}, so(m)) \hookrightarrow L^{\frac{2r}{2-r}}$, we also have

$$\|\eta P_0^{-1}\Delta^{1/4}P_0\|_{L^r} \leq C \|\eta\|_{\dot{W}^{1/2,r}} \|P_0^{-1}\Delta^{1/4}P_0\|_{L^2}. \quad (52)$$

We consider the following map $H^{P_0}: \dot{W}^{1/2,r}(\mathbb{R}, so(m)) \rightarrow L^r(\mathbb{R}, so(m))$,

$$H^{P_0}(\eta) = -2\eta P_0^{-1}\Delta^{1/4}P_0 - 2\Delta^{1/4}P_0^{-1}P_0\eta + Q(\eta, P_0) - Q^t(\eta, P_0).$$

From (50) and (52), it follows that there exists a constant $C > 0$ (independent of P_0) such that

$$\|H^{P_0}(\eta)\|_{L^r} \leq C \|\eta\|_{\dot{W}^{1/2,r}} \left[\|P_0\|_{\dot{H}^{1/2}} + \|P_0\|_{\dot{H}^{1/2}}^2 \|P_0\|_{L^\infty} \right].$$

Because of (46), $\|P_0\|_{\dot{H}^{1/2}} \leq (C\varepsilon)^{1/2}$ and hence, if $\varepsilon > 0$ is small enough, $L^{P_0} = 2\Delta^{1/4} + H_{P_0}: \dot{W}^{1/2,r}(\mathbb{R}, so(m)) \rightarrow L^r(\mathbb{R}, so(m))$ is invertible which proves the first claim.

Claim 2. Let $q' < r < 2$. Let $\omega \in L^{q'} \cap L^r$ and η be the solution of $L^{P_0}(\eta) = \omega$ then η is in $\dot{W}^q \cap \dot{W}^r$.

We apply Lemma B.7 to

$$\Delta^{1/4}\eta - P_0^{-1}\Delta^{1/4}(P_0\eta) = \Delta^{1/4}(P_0^{-1}P_0\eta) - P_0^{-1}\Delta^{1/4}(P_0\eta)$$

and we obtain

$$\begin{aligned} \|\Delta^{1/4}\eta - P_0^{-1}\Delta^{1/4}(P_0\eta)\|_{L^t} &\leq \|P_0\eta\|_{\dot{W}^{1/2,r}(\mathbb{R},so(m))} \|P_0\|_{\dot{W}^{1/2,q}(\mathbb{R},so(m))} \\ &\leq \|\eta\|_{\dot{W}^{1/2,r}} [\|P_0\|_{L^\infty} + \|P_0\|_{\dot{H}^{1/2}}] \|P_0\|_{\dot{W}^{1/2,q}(\mathbb{R},so(m))}, \end{aligned} \quad (53)$$

where t is given by $\frac{1}{t} = \frac{1}{q} + \frac{2-r}{2r}$. In a similar way we have

$$\|\Delta^{1/4}\eta - \Delta^{1/4}(\eta P_0^{-1})P_0\|_{L^t} \leq \|\eta\|_{\dot{W}^{1/2,r}} [\|P_0\|_{L^\infty} + \|P_0\|_{\dot{H}^{1/2}}] \|P_0\|_{\dot{W}^{1/2,q}(\mathbb{R},so(m))}.$$

On the other hand we also have

$$\|\eta P_0^{-1}\Delta^{1/4}P_0\|_{L^t} \leq \|\eta\|_{L^{\frac{2r}{2-r}}} \|\Delta^{1/4}P_0\|_{L^q} \quad (54)$$

Thus $Q(\eta, P_0)$, $Q^t(\eta, P_0)$ and $H_{P_0}(\eta)$ are in L^t . Thus since $\omega \in L^q \cap L^r$, we have $\Delta^{1/4}\eta \in L^t$ as well. Since $q' < r < 2$ and $\frac{1}{t} = \frac{1}{q} + \frac{1}{r} - \frac{1}{2}$, we have that $t > 2$. $\Delta^{1/4}\eta \in L^r \cap L^t$ for some $r < 2$ and $t > 2$ implies that $\eta \in L^\infty$ (see for instance [2], pag 25).

From the fact that $\eta \in L^\infty$ we deduce that $\eta P_0^{-1}\Delta^{1/4}P_0 \in L^q$ and $\Delta^{1/4}P_0^{-1}P_0\eta \in L^q$. Now we apply Lemma B.7 respectively to $a = P_0\eta \in \dot{H}^{1/2} \cap L^\infty$, $b = P_0^{-1} \in \dot{W}^{1/2,q}$ and $a = \eta P_0^{-1}$, $b = P_0$ and we get that $H_{P_0}(\eta) \in L^q$. Since $\omega \in L^q \cap L^r$ we have $\Delta^{1/4}\eta \in L^q$ as well. Moreover the following estimate holds

$$\|\Delta^{1/4}\eta\|_{L^q} \leq C \|\omega\|_{L^q \cap L^r} \leq C \|\omega\|_{L^q \cap L^{q'}},$$

which proves the claim 2.

Combining claim 1 and claim 2 we obtain that for any $\omega \in L^q \cap L^{q'}(\mathbb{R}, so(m))$ there exists a unique $\eta \in \dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}(\mathbb{R}, so(m))$ such that

$$L^{P_0}\eta = \omega,$$

and

$$\|\eta\|_{\dot{W}^{1/2,q} \cap \dot{W}^{1/2,q'}} \leq C \|\omega\|_{L^q \cap L^{q'}}$$

This finishes the proof of lemma 4.1. □

Proof of step 2 continued. We apply Implicit Function Theorem to F^{P_0} and we deduce that for every P in some neighborhood of P_0 and Ω in a neighborhood of Ω_0 (both neighborhoods having a size depending on P_0 and Ω_0 of course) the equation (8)(i) is satisfied and for some constant $C > 0$

$$\|\Delta^{1/4}P\|_{L^q} \leq C\|\Omega\|_{L^q}.$$

By possibly taking a smaller neighborhood of P_0 we may always assume that $\int_{\mathbb{R}} |\Delta^{1/4}P|^2 dx \leq \varepsilon < 1$.

Step 3: *The fact that $\int_{\mathbb{R}} |\Delta^{1/4}P|^2 dx \leq \varepsilon < 1$ implies that $\int_{\mathbb{R}} |\Delta^{1/4}P|^2 dx \leq C \int_{\mathbb{R}} |\Omega|^2 dx$.*

We write

$$\begin{aligned} P^{-1}\Delta^{1/4}P &= \frac{1}{2}(P^{-1}\Delta^{1/4}P - \Delta^{1/4}(P^{-1}\Delta^{1/4}P)^t) + \frac{1}{2}(P^{-1}\Delta^{1/4}P + (P^{-1}\Delta^{1/4}P)^t) \\ &= \frac{1}{2}(P^{-1}\Delta^{1/4}P - \Delta^{1/4}P^{-1}P) + \frac{1}{2}(P^{-1}\Delta^{1/4}P + \Delta^{1/4}P^{-1}P). \end{aligned}$$

We set

$$\text{symm}(P^{-1}\Delta^{1/4}P) := \frac{1}{2}(P^{-1}\Delta^{1/4}P + \Delta^{1/4}P^{-1}P)$$

and

$$\text{Asymm}(P^{-1}\Delta^{1/4}P) := \frac{1}{2}(P^{-1}\Delta^{1/4}P - \Delta^{1/4}P^{-1}P).$$

We apply Lemma B.5 and we get

$$\begin{aligned} \int_{\mathbb{R}} |P^{-1}\Delta^{1/4}P + \Delta^{1/4}P^{-1}P|^2 dx &\leq C\|P^{-1}\Delta^{1/4}P\|_{L^2}^2 \\ &\leq C\|\Delta^{1/4}P\|_{L^2} (\|\text{symm}(P^{-1}\Delta^{1/4}P)\|_{L^2} + \|\text{Asymm}(P^{-1}\Delta^{1/4}P)\|_{L^2}). \end{aligned}$$

Thus we get

$$\|\text{symm}(P^{-1}\Delta^{1/4}P)\|_{L^2} \leq C\varepsilon (\|\text{symm}(P^{-1}\Delta^{1/4}P)\|_{L^2} + \|\text{Asymm}(P^{-1}\Delta^{1/4}P)\|_{L^2}).$$

If $\varepsilon > 0$ is small enough then

$$\|\text{symm}(P^{-1}\Delta^{1/4}P)\|_{L^2} \leq C\|\text{Asymm}(P^{-1}\Delta^{1/4}P)\|_{L^2} = C\|\Omega\|_{L^2}$$

which ends the proof of Step 3.

Step 4. Take now $\Omega \in L^2$ and $\int_{\mathbb{R}} |\Omega|^2 dx \leq \varepsilon$. Let $\Omega_k \in \mathcal{U}_\varepsilon^q$ be such that $\Omega_k \rightarrow \Omega$ as $k \rightarrow +\infty$ in L^2 . By arguing as in the proof of that $\mathcal{V}_\varepsilon^q$ is closed one gets that there exists $P \in \dot{H}^{1/2}$ satisfying (8)(i)-(ii). \square

5 Euler Equation for Half-Harmonic Maps into Manifolds

We consider a compact k dimensional C^2 manifold without boundary $\mathcal{N} \subset \mathbb{R}^m$. Let $\Pi_{\mathcal{N}}$ be the orthogonal projection on \mathcal{N} . We also consider the Dirichlet energy (13).

The weak 1/2-harmonic maps are defined as critical points of the functional (13) with respect to perturbation of the form $\Pi_{\mathcal{N}}(u + t\phi)$, where ϕ is an arbitrary compacted supported smooth map from \mathbb{R} into \mathbb{R}^m .

Definition 5.1 We say that $u \in H^{1/2}(\mathbb{R}, \mathcal{N})$ is a weak 1/2-harmonic map if and only if, for every maps $\phi \in H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ we have

$$\frac{d}{dt} \mathcal{L}(\Pi_{\mathcal{N}}(u + t\phi))|_{t=0} = 0. \quad (55)$$

□

We introduce some notations. We denote by $\bigwedge(\mathbb{R}^m)$ the exterior algebra (or Grassmann Algebra) of \mathbb{R}^m and by the symbol \wedge the *exterior or wedge product*. For every $p = 1, \dots, m$, $\bigwedge_p(\mathbb{R}^m)$ is the vector space of p -vectors

If $(\epsilon_i)_{i=1, \dots, m}$ is the canonical orthonormal basis of \mathbb{R}^m , then every element $v \in \bigwedge_p(\mathbb{R}^m)$ is written as $v = \sum_I v_I \epsilon_I$ where $I = \{i_1, \dots, i_p\}$ with $1 \leq i_1 \leq \dots \leq i_p \leq m$, $v_I := v_{i_1, \dots, i_p}$ and $\epsilon_I := \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_p}$.

By the symbol L we denote the interior multiplication $\mathsf{L}: \bigwedge_p(\mathbb{R}^m) \times \bigwedge_q(\mathbb{R}^m) \rightarrow \bigwedge_{q-p}(\mathbb{R}^m)$ defined as follows.

Let $\epsilon_I = \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_p}$, $\epsilon_J = \epsilon_{j_1} \wedge \dots \wedge \epsilon_{j_q}$, with $q \geq p$. Then $\epsilon_I \mathsf{L} \epsilon_J = 0$ if $I \not\subset J$, otherwise $\epsilon_I \mathsf{L} \epsilon_J = (-1)^M \epsilon_K$ where ϵ_K is a $q - p$ vector and M is the number of pairs $(i, j) \in I \times J$ with $j > i$.

Finally by the symbol $*$ we denote the Hodge-star operator, $*$: $\bigwedge_p(\mathbb{R}^m) \rightarrow \bigwedge_{m-p}(\mathbb{R}^m)$, defined by $*\beta = \beta \mathsf{L}(\epsilon_1 \wedge \dots \wedge \epsilon_n)$. For an introduction of the Grassmann Algebra we refer the reader to the first Chapter of the book by Federer[6].

In the sequel we denote by P^T and P^N respectively the tangent and the normal projection to the manifold \mathcal{N} .

They verify the following properties: $(P^T)^t = P^T$, $(P^N)^t = P^N$ (namely they are symmetric operators), $(P^T)^T = P^T$, $(P^N)^N = P^N$, $P^T + P^N = Id$, $P^N P^T = P^T P^N = 0$.

We set $e = \epsilon_1 \wedge \dots \wedge \epsilon_k$ and $n = \epsilon_{k+1} \wedge \dots \wedge \epsilon_m$. For every $z \in \mathcal{N}$, $e(z)$ and $n(z)$ give the orientation respectively of the tangent k -plane and the normal $m - k$ -plane to $T_z \mathcal{N}$.

We observe that for every $v \in \mathbb{R}^m$ we have

$$P^T v = (-1)^{m-1} * ((v \mathsf{L} e) \wedge n). \quad (56)$$

$$P^N v = (-1)^{k-1} * (e \wedge (v \mathsf{L} n)). \quad (57)$$

We observe that P^N and P^T can be seen as matrices in $\dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$.

Next we write the Euler equation associated to the functional (13).

Proposition 5.1 All weak 1/2-harmonic maps $u \in H^{1/2}(\mathbb{R}, \mathcal{N})$ satisfy in a weak sense i) the equation

$$\int_{\mathbb{R}} (\Delta^{1/2} u) \cdot v \, dx = 0, \quad (58)$$

for every $v \in H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ and $v \in T_{u(x)} \mathcal{N}$ almost everywhere, or in a equivalent way

ii) the equation

$$P^T \Delta^{1/2} u = 0 \quad \text{in } \mathcal{D}', \quad (59)$$

or

iii) the equation

$$\Delta^{1/4}(P^T \Delta^{1/4} u) = T(P^T, u) - (\Delta^{1/4} P^T) \Delta^{1/4} u, \quad (60)$$

□

The Euler Lagrange equation (60) can be completed by the following "structure equation":

Proposition 5.2 *All maps in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ satisfy the following identity*

$$\Delta^{1/4}(\mathcal{R}(P^N \Delta^{1/4} u)) = \mathcal{R}(S(P^N, u)) - (\Delta^{1/4} P^N)(\mathcal{R} \Delta^{1/4} u). \quad (61)$$

□

For the proofs of Proposition 5.1 and 5.2 we refer the reader to [4].

Next we see that by combining (60) and (61) we can obtain the new equation (1.1) for the vector field $v = (P^T \Delta^{1/4} u, \mathcal{R}(P^N \Delta^{1/4} u))$ where an antisymmetric potential appears.

We introduce the following matrices

$$\omega_1 = \frac{(\Delta^{1/4} P^T) P^T + P^T \Delta^{1/4} P^T - \Delta^{1/4} (P^T P^T)}{2}, \quad (62)$$

$$\omega_2 = (\Delta^{1/4} P^T) P^N + P^T \Delta^{1/4} P^N - \Delta^{1/4} (P^T P^N), \quad (63)$$

$$\omega = \frac{(\Delta^{1/4} P^T) P^T - P^T \Delta^{1/4} P^T}{2}; \quad (64)$$

and

$$\omega_3 = \frac{(\mathcal{R} \Delta^{1/4} P^T) P^T + P^T \Delta^{1/4} (\mathcal{R} \Delta^{1/4} P^T) - \mathcal{R} \Delta^{1/4} (P^T P^T)}{2}, \quad (65)$$

$$\omega_4 = (\mathcal{R} \Delta^{1/4} P^T) P^N + P^N (\mathcal{R} \Delta^{1/4} P^T) - \mathcal{R} \Delta^{1/4} (P^N P^T), \quad (66)$$

$$\omega_{\mathcal{R}} = \frac{(\mathcal{R} \Delta^{1/4} P^T) P^T - P^T (\mathcal{R} \Delta^{1/4} P^T)}{2}. \quad (67)$$

We observe that Theorem 1.3 and Theorem 1.4 imply respectively that $\Delta^{1/4}(\omega_1)$, $\Delta^{1/4}(\omega_2)$ and $\Delta^{1/4}(\omega_3)$, $\Delta^{1/4}(\omega_4)$ are in the homogeneous Hardy Space $\mathcal{H}^1(\mathbb{R})$. Therefore $\omega_1, \omega_2, \omega_3, \omega_4 \in L^{2,1}(\mathbb{R})$. The matrices ω and $\omega_{\mathcal{R}}$ are **antisymmetric**.

Proof of Proposition 1.1. From Propositions 5.1 and 5.2 it follows that u satisfies in a weak sense the equations (60) and (61).

The key point is to estimate the terms $(\Delta^{1/4} P^T) \Delta^{1/4} u$ and $(\Delta^{1/4} P^N) \mathcal{R}(\Delta^{1/4} u)$

• **Re-writing of $(\Delta^{1/4} P^T) \Delta^{1/4} u$.**

$$\begin{aligned}
(\Delta^{1/4} P^T) \Delta^{1/4} u &= (\Delta^{1/4} P^T) (P^T \Delta^{1/4} u + P^N \Delta^{1/4} u) \\
&= ((\Delta^{1/4} P^T) P^T) (P^T v) + ((\Delta^{1/4} P^T) P^N) (P^N v).
\end{aligned}$$

Now we have

$$(\Delta^{1/4} P^T) P^T = \omega_1 + \omega + \frac{\Delta^{1/4} P^T}{2}; \quad (68)$$

and

$$\begin{aligned}
(\Delta^{1/4} P^T) P^N &= (\Delta^{1/4} P^T) P^N + P^T \Delta^{1/4} P^N - \Delta^{1/4} (P^T P^N) - P^T \Delta^{1/4} P^N \\
&= \omega_2 + P^T \Delta^{1/4} P^T \\
&= \omega_2 + \omega_1 - \omega + \frac{\Delta^{1/4} P^T}{2}.
\end{aligned} \quad (69)$$

Thus

$$\frac{(\Delta^{1/4} P^T) (P^T \Delta^{1/4} u)}{2} = \omega_1 (P^T \Delta^{1/4} u) + \omega (P^T \Delta^{1/4} u) \quad (70)$$

$$\frac{(\Delta^{1/4} P^T) (P^N \Delta^{1/4} u)}{2} = (\omega_1 + \omega_2) (P^N \Delta^{1/4} u) - \omega (P^N \Delta^{1/4} u) \quad (71)$$

$$\begin{aligned}
&= \mathcal{R}(\omega_1 + \omega_2) \mathcal{R}(P^N \Delta^{1/4} u) - \mathcal{R}(\omega) \mathcal{R}(P^N \Delta^{1/4} u) \\
&+ F(-\omega + \omega_1 + \omega_2, (P^N \Delta^{1/4} u)).
\end{aligned}$$

• **Re-writing of $(\Delta^{1/4} P^N) (\mathcal{R} \Delta^{1/4} u)$.**

We have

$$\begin{aligned}
(\Delta^{1/4} P^N) (\mathcal{R} \Delta^{1/4} u) &= (\mathcal{R}(\Delta^{1/4} P^N)) (P^T (\Delta^{1/4} u) + P^N (\Delta^{1/4} u)) \\
&+ F((\mathcal{R}(\Delta^{1/4} P^N)), \Delta^{1/4} u).
\end{aligned}$$

We estimate $(\mathcal{R} \Delta^{1/4} P^N) P^T (\Delta^{1/4} u)$ and $(\mathcal{R} \Delta^{1/4} P^N) P^N (\Delta^{1/4} u)$. We have

$$\begin{aligned}
(\mathcal{R} \Delta^{1/4} P^N) P^T &= -(\mathcal{R} \Delta^{1/4} P^T) P^T \\
&= -\omega_3 - \omega_{\mathcal{R}} - \frac{(\mathcal{R} \Delta^{1/4} P^T)}{2} \\
&= -\omega_3 - \omega_{\mathcal{R}} + \frac{(\mathcal{R} \Delta^{1/4} P^N)}{2},
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{R}\Delta^{1/4}P^N)P^N &= -(\mathcal{R}\Delta^{1/4}P^T)P^N \pm P^T(\mathcal{R}\Delta^{1/4}P^N) \\
&= -[(\mathcal{R}\Delta^{1/4}P^T)P^N + P^T(\mathcal{R}\Delta^{1/4}P^N) - \mathcal{R}\Delta^{1/4}(P^N P^T)] \\
&\quad + P^T(\mathcal{R}\Delta^{1/4}P^N) \\
&= -\omega_4 - \omega_3 + \omega_{\mathcal{R}} + \frac{(\mathcal{R}\Delta^{1/4}P^N)}{2}.
\end{aligned}$$

Thus

$$\frac{(\mathcal{R}\Delta^{1/4}P^N)P^T\Delta^{1/4}u}{2} = -\omega_3(P^T\Delta^{1/4}u) - \omega_{\mathcal{R}}(P^T\Delta^{1/4}u) \quad (72)$$

$$\begin{aligned}
\frac{(\mathcal{R}\Delta^{1/4}P^N)P^N\Delta^{1/4}u}{2} &= -\omega_4(P^N\Delta^{1/4}u) - \omega_3(P^N\Delta^{1/4}u) + \omega_{\mathcal{R}}(P^N\Delta^{1/4}u) \quad (73) \\
&= \mathcal{R}(-\omega_3 - \omega_4)\mathcal{R}(P^N\Delta^{1/4}u) \\
&\quad + \mathcal{R}(\omega_{\mathcal{R}})\mathcal{R}(P^N\Delta^{1/4}u) \\
&\quad + F(\omega_{\mathcal{R}} - \omega_3 - \omega_4, P^N\Delta^{1/4}u).
\end{aligned}$$

By combining (70), (71), (72), (73) we obtain

$$\begin{aligned}
\Delta^{1/4} \begin{pmatrix} P^T\Delta^{1/4}u \\ \mathcal{R}P^N\Delta^{1/4}u \end{pmatrix} &= \tilde{\Omega}_1 + \tilde{\Omega}_2 \begin{pmatrix} P^T\Delta^{1/4}u \\ \mathcal{R}P^N\Delta^{1/4}u \end{pmatrix} \quad (74) \\
&\quad + 2 \begin{pmatrix} -\omega & \omega_{\mathcal{R}} \\ \omega_{\mathcal{R}} & -\mathcal{R}\omega_{\mathcal{R}} \end{pmatrix} \begin{pmatrix} P^T\Delta^{1/4}u \\ \mathcal{R}P^N\Delta^{1/4}u \end{pmatrix},
\end{aligned}$$

where $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given by

$$\begin{aligned}
\tilde{\Omega}_1 &= \begin{pmatrix} -2F(-\omega + \omega_1 + \omega_2, (P^N\Delta^{1/4}u)) + T(P^T, u) \\ -2F(\mathcal{R}(\Delta^{1/4}P^N), \mathcal{R}(\Delta^{1/4}u)) - 2F(\omega_{\mathcal{R}} - \omega_3 - \omega_4, P^N(\Delta^{1/4}u) + \mathcal{R}(S(P^N, u))) \end{pmatrix}. \\
\tilde{\Omega}_2 &= 2 \begin{pmatrix} -\omega_1 & -[\mathcal{R}(\omega_1 + \omega_2) + (\mathcal{R}(\omega) - \omega_{\mathcal{R}})] \\ \omega_3 & -\mathcal{R}(\omega_3 - \omega_4) \end{pmatrix}.
\end{aligned}$$

The matrix

$$\Omega = 2 \begin{pmatrix} -\omega & \omega_{\mathcal{R}} \\ \omega_{\mathcal{R}} & -\mathcal{R}\omega_{\mathcal{R}} \end{pmatrix}$$

is antisymmetric.

We observe that from the estimate (18) it follows that $\tilde{\Omega}_1 \in \mathcal{H}^{-1/2}(\mathbb{R}, \mathbb{R}^{2m})$ and

$$\|\tilde{\Omega}_1\|_{H^{-1/2}(\mathbb{R})} \leq C(\|P^N\|_{\dot{H}^{1/2}(\mathbb{R})} + \|P^T\|_{\dot{H}^{1/2}(\mathbb{R})})\|\Delta^{1/4}u\|_{L^{2,\infty}}. \quad (75)$$

On the other hand $\tilde{\Omega}_2 \in L^{2,1}(\mathbb{R}, \mathcal{M}_{2m})$ and

$$\|\tilde{\Omega}_2\|_{L^{2,1}(\mathbb{R})} \leq C(\|P^N\|_{\dot{H}^{1/2}(\mathbb{R})}^2 + \|P^T\|_{\dot{H}^{1/2}(\mathbb{R})}^2). \quad (76)$$

This concludes the proof of proposition 1.1. \square

Proof of Theorem 1.7.

From Proposition 1.1 it follows that $v = (P^T(\Delta^{1/4}u), \mathcal{R}(P^N(\Delta^{1/4}u)))$ solves equation (74) which is of the type (5) up to the term $\tilde{\Omega}_1$. Therefore the arguments are very similar to those of Theorem 1.1 and we give only a sketch of proof.

We aim at obtaining that $\Delta^{1/4}u \in L_{loc}^p(\mathbb{R})$, for all $p \geq 1$. To this purpose we take $\rho > 0$ such that

$$\|\Omega\|_{L^2(B(0,\rho))}, \|P^T\|_{\dot{H}^{1/2}(B(0,\rho))}, \|P^N\|_{\dot{H}^{1/2}(B(0,\rho))} \leq \varepsilon_0,$$

with $\varepsilon_0 > 0$ small enough. Let $x_0 \in B(0, \rho/4)$ and $r \in (0, \rho/8)$. As in the case of equation (5) we argue by duality and multiply both sides of equation (74) by $\phi = \Delta^{-1/4}(g_{r\alpha})$, with $g \in L^{2,1}(\mathbb{R})$, $\|g\|_{L^{2,1}} \leq 1$ and $g_{r\alpha} = \mathbb{1}_{B(x_0, r\alpha)}g$, with $0 < \alpha < 1/4$.

It is enough to estimate the integral

$$\int_{\mathbb{R}} \tilde{\Omega}_1 \left(\begin{array}{c} \Delta^{-1/4}P^T(g_{r\alpha}) \\ \Delta^{-1/4}P^N(g_{r\alpha}) \end{array} \right) dx. \quad (77)$$

(being the other terms already estimated in the proof of Theorem 1.1).

We observe that

$$\|\Delta^{1/4}u\|_{L^{2,\infty}} \lesssim \left\| \sqrt{(P^T(\Delta^{1/4}u))^2 + (\mathcal{R}(P^N(\Delta^{1/4}u)))^2} \right\|_{L^{2,\infty}} = \|v\|_{L^{2,\infty}} \quad (78)$$

By combining Lemma A.1, A.2, A.5 and A.6 and the estimate (78) we obtain

$$\begin{aligned} (77) &\lesssim \varepsilon_0 \|\Delta^{1/4}u\|_{L^{2,\infty}} + \alpha^{1/2} \sum_{h=1}^{+\infty} 2^{-h/2} \|\Delta^{1/4}u\|_{L^{2,\infty}(B_{2^{h+1}r} \setminus B_{2^h r})} \\ &\lesssim \varepsilon_0 \|v\|_{L^{2,\infty}} + \alpha^{1/2} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(B_{2^{h+1}r} \setminus B_{2^h r})}. \end{aligned}$$

Therefore v satisfies an estimate of the type (36) which implies $\|v\|_{L^{2,\infty}(B(x_0, r))} \leq Cr^\beta$, for α and ε_0 small enough, for all $x_0 \in B(0, \rho/4)$ and $0 < r < \rho/8$ and for some $\beta \in (0, 1/2)$.

By arguing as in Theorem 1.1 we deduce that $v \in L_{loc}^p(\mathbb{R})$, for all $p \geq 1$. Therefore $\Delta^{1/4}u \in L_{loc}^p(\mathbb{R})$, for all $p \geq 1$ as well.

This implies that $u \in C_{loc}^{0,\alpha}$ for some $0 < \alpha < 1$, since $W_{loc}^{1/2,p}(\mathbb{R}) \hookrightarrow C_{loc}^{0,\alpha}(\mathbb{R})$ if $p > 2$ (see for instance [2]). This concludes the proof of Theorem 1.7. \square

A Localization Estimates

The aim of this Appendix is to provide localization estimates for the terms appearing in the equation (5) and the equation (74).

For $r > 0$, $h \in \mathbb{Z}$ and $x_0 \in \mathbb{R}$ we set

$$A_{h,x_0} = B(x_0, 2^{h+1}) \setminus B(x_0, 2^h) \quad \text{and} \quad A'_{h,x_0} = B(x_0, 2^h) \setminus B(x_0, 2^{h-1}).$$

We first localize the term $N(Q, v)$.

Lemma A.1 *Let $Q \in \dot{H}^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \leq \varepsilon_0$, $v \in L^2(\mathbb{R})$, $g \in L^{2,1}(\mathbb{R})$, $\text{supp } g \subset B(x_0, r\alpha)$, with $x_0 \in \mathbb{R}$, $0 < \alpha < \frac{1}{4}$, $r > 0$. Then we have*

$$\begin{aligned} \int_{\mathbb{R}} N(Q, v) \Delta^{-1/4} g dx &\lesssim \varepsilon_0 \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0, r))} \\ &+ (\|Q\|_{\dot{H}^{1/2}(\mathbb{R})} + \|Q\|_{L^\infty}) \|g\|_{L^{2,1}} \alpha^{1/2} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0})}. \end{aligned} \quad (79)$$

Proof of Lemma A.1. We consider a dyadic decomposition of the unity $\varphi_j \in C_0^\infty(\mathbb{R})$ such that

$$\text{supp}(\varphi_j) \subset B_{2^{j+1}r}(x_0) \setminus B_{2^{j-1}r}(x_0), \quad \sum_{-\infty}^{+\infty} \varphi_j = 1. \quad (80)$$

We set $\chi_r := \sum_{-\infty}^0 \varphi_j$. We observe that the function $\psi = \Delta^{-1/4} g$ is in $L^\infty(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})$. We take the scalar product of $N(Q, v)$ with $\Delta^{-1/4} g$ and we integrate. We write

$$\begin{aligned} \int_{\mathbb{R}} N(Q, v) \Delta^{-1/4} g dx &= \underbrace{\int_{\mathbb{R}} N(Q, \chi_r v) \Delta^{-1/4} g dx}_{(1)} \\ &+ \underbrace{\int_{\mathbb{R}} \sum_{h=1}^{+\infty} N(Q, \varphi_h v) \Delta^{-1/4} g dx}_{(2)} \end{aligned}$$

To estimate (1) we use the fact that $N(Q, v) \in \dot{H}^{-1/2}(\mathbb{R})$ and

$$\|N(Q, v)\|_{\dot{H}^{-1/2}(\mathbb{R})} \lesssim \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}(\mathbb{R})}.$$

$$\begin{aligned} (1) &\leq \|\Delta^{-1/4} g\|_{\dot{H}^{1/2}(\mathbb{R})} \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}} \\ &\lesssim \varepsilon_0 \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0, r))}. \end{aligned}$$

Next we split (2) in two parts:

$$\begin{aligned}
(2) &= \underbrace{\sum_{k=1}^{\infty} \int_{\mathbb{R}} N(Q, \varphi_k v) \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g dx}_{(3)} \\
&+ \underbrace{\sum_{k=1}^{\infty} \sum_{h=-1}^{\infty} \int_{\mathbb{R}} N(Q, \varphi_k v) \mathbb{1}_{A'_{h, x_0}} \Delta^{-1/4} g dx}_{(4)}.
\end{aligned}$$

We observe that in (3) and (4) we can exchange the integral with the infinite sum (see the Appendix in [4]).

We estimate (3). We first observe that since $\mathbb{1}_{B(x_0, r/4)}$ and φ_k have disjoint supports, we have

$$N(Q, \varphi_k v) \mathbb{1}_{B(x_0, r/4)} = [\Delta^{1/4}(Q\varphi_k v) - Q\Delta^{1/4}(\varphi_k v)] \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g dx.$$

Thus

$$\begin{aligned}
(3) &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} [\Delta^{1/4}(Q\varphi_k v) - Q\Delta^{1/4}(\varphi_k v)] \mathbb{1}_{B(x_0, r/4)} dx \\
&\simeq \sum_{k=1}^{\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}(|\cdot|^{1/2})(\xi) \\
&\quad [Q(\varphi_k v) * (\mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g) - (\varphi_k v) * (Q\mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g)] d\xi \\
&\lesssim \sum_{k=1}^{\infty} \|\xi|^{-3/2}\|_{L^\infty(B^c(0, 2^h r))} \\
&\quad [\|Q(\varphi_k v) * (\mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g)\|_{L^1(\mathbb{R})} + \|(\varphi_k v) * (Q\mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g)\|_{L^1}] \\
&\lesssim \sum_{k=1}^{\infty} 2^{-3/2k} r^{-3/2} \left[2\|Q\|_{L^\infty} \|\varphi_k v\|_{L^1(A_{h, x_0})} \|\mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g\|_{L^1(\mathbb{R})} \right] \\
&\lesssim \sum_{k=1}^{\infty} 2^{-3/2k} r^{-3/2} 2^{k/2} r^{1/2} r^{1/2} (r\alpha)^{1/2} \left[\|Q\|_{L^\infty} \|v\|_{L^\infty(A_{h, x_0})} \|g\|_{L^{2,1}(\mathbb{R})} \right] \\
&\lesssim \alpha^{1/2} \|Q\|_{L^\infty(\mathbb{R})} \|g\|_{L^{2,1}(\mathbb{R})} \sum_{k=1}^{\infty} 2^{-k/2} \|v\|_{L^\infty(A_{k, x_0})}.
\end{aligned}$$

We estimate (4).

$$\begin{aligned}
(4) &= \underbrace{\sum_{k=1}^{\infty} \sum_{|k-h| \leq 5} \int_{\mathbb{R}} N(Q, \varphi_k v) \mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g dx}_{(5)} \\
&= \underbrace{\sum_{k=1}^{\infty} \sum_{|k-h| \geq 5} \int_{\mathbb{R}} N(Q, \varphi_k v) \mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g dx}_{(6)}.
\end{aligned}$$

We observe that

$$\|\mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g\|_{\dot{H}^{1/2}(\mathbb{R})} \lesssim \|g\|_{L^{2,1}(\mathbb{R})} \alpha^{1/2} 2^{-h/2}.$$

Thus

$$\begin{aligned}
(5) &\leq \sum_{k=1}^{\infty} \sum_{|k-h| \geq 5} \|N(Q, \varphi_k v)\|_{\dot{H}^{-1/2}(\mathbb{R})} \|\mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g\|_{\dot{H}^{1/2}(\mathbb{R})} \\
&\lesssim \alpha^{1/2} \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|g\|_{L^{2,1}(\mathbb{R})} \sum_{k=1}^{\infty} 2^{-k/2} \|v\|_{L^{2,\infty}(A_{k,x_0})}.
\end{aligned}$$

In order to estimate (6) we observe if $|k-h| \geq 6$ then $\varphi_k v$ and $\mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g$ have disjoint supports. Thus by arguing as in (3) we get

$$(6) \lesssim \alpha^{1/2} \|Q\|_{L^\infty} \|g\|_{L^{2,1}(\mathbb{R})} \sum_{k=1}^{+\infty} 2^{-k/2} \|v\|_{L^{2,\infty}(A_{k,x_0})}.$$

This concludes the proof of Lemma A.1. \square

Lemma A.2 *Let $Q \in \dot{H}^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\text{supp } Q \subset B^c(0, \rho)$ for some $\rho > 0$, $v \in L^2(\mathbb{R})$, $x_0 \in B(0, \rho/4)$, $g \in L^{2,1}(\mathbb{R})$, $\text{supp } g \subset B(x_0, r\alpha)$, with $0 < \alpha < 1$, $0 < r < \rho/8$.*

Then we have

$$\begin{aligned}
&\int_{\mathbb{R}} N(Q, v) \Delta^{-1/4} g dx \lesssim \left(\frac{r}{\rho}\right)^{1/2} \|g\|_{L^{2,1}(\mathbb{R})} \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}(B(x_0, r))} \\
&+ \alpha^{1/2} (\|Q\|_{L^\infty} + \|Q\|_{\dot{H}^{1/2}(\mathbb{R})}) \|g\|_{L^{2,1}(\mathbb{R})} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0})}.
\end{aligned} \tag{81}$$

Proof of Lemma A.2. We write

$$\begin{aligned}
\int_{\mathbb{R}} N(Q, v) \Delta^{-1/4} g dx &= \underbrace{\int_{\mathbb{R}} N(Q, \chi_r v) \Delta^{-1/4} g dx}_{(7)} \\
&= \underbrace{\int_{\mathbb{R}} N(Q, (1 - \chi_r) v) \Delta^{-1/4} g dx}_{(8)}.
\end{aligned}$$

We denote by $Q_\rho = |B_\rho(0)|^{-1} \int_{B_\rho(0)} Q(y) dy = 0$ and write $Q = \sum_{h=-1}^{+\infty} \tilde{\varphi}_h(Q - Q_\rho)$, with $\text{supp}(\tilde{\varphi}_h) \subset B(0, 2^{h+1}\rho) \setminus B(0, 2^h\rho)$, $\tilde{\varphi}$ partition of unity.

We estimate (7).

$$\begin{aligned}
(7) &= \int_{\mathbb{R}} N\left(\sum_{h=-1}^{+\infty} \varphi_h(Q - Q_\rho), \chi_r v\right) \Delta^{-1/4} g dx \\
&= \sum_{h=-1}^{+\infty} \int_{\mathbb{R}} [-\varphi_h(Q - Q_\rho) \Delta^{1/4} (\chi_r v) \Delta^{-1/4} g + \Delta^{1/4} (\varphi_h(Q - Q_\rho) (\chi_r v) \Delta^{-1/4} g)] dx \\
&= \sum_{h=-1}^{+\infty} \mathcal{F}^{-1}[|\cdot|^{1/2}](\xi) \\
&\quad \left[-(\chi_r v) * (\varphi_h(Q - Q_\rho) \Delta^{-1/4} g) + \varphi_h(Q - Q_\rho) * (\chi_r v \Delta^{-1/4} g) \right] dx \\
&\lesssim \sum_{h=-1}^{+\infty} \| |\xi|^{-3/2} \|_{L^\infty(B^c(0, 2^h\rho))} \left[\|\chi_r v\|_{L^1} \|\varphi_h(Q - Q_\rho)\|_{L^1} \|\Delta^{-1/4} g\|_{L^\infty} \right] \\
&\lesssim \|g\|_{L^{2,1}(\mathbb{R})} \sum_{h=-1}^{+\infty} 2^{-h/2} \left(\frac{r}{\rho}\right)^{1/2} \|v\|_{L^{2,\infty}(B(x_0, r))} \|\varphi_h(Q - Q_\rho)\|_{\dot{H}^{1/2}(\mathbb{R})} \\
&\text{by Lemma 4.1 in [4]} \\
&\lesssim \left(\frac{r}{\rho}\right)^{1/2} \|g\|_{L^{2,1}(\mathbb{R})} \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}(B(x_0, r))}.
\end{aligned}$$

By arguing as in (3) and (4) we get

$$(8) \lesssim (\|Q\|_{L^\infty} + \|Q\|_{\dot{H}^{1/2}(\mathbb{R})}) \|g\|_{L^{2,1}(\mathbb{R})} \alpha^{1/2} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0})}. \quad (82)$$

This concludes the proof of Lemma A.2. \square

The localization of the operator $S(Q, \Delta^{-1/4} v)$, with $v \in L^2(\mathbb{R})$ is similar to that of $N(Q, v)$ and we omit it.

Lemma A.3 *Let $A \in L^{2,1}(\mathbb{R})$, $x_0 \in \mathbb{R}$, $r > 0$, $0 < \alpha < 1$ and $g \in L^{2,1}(\mathbb{R})$, $\text{supp } g \subset B(x_0, r\alpha)$. Then*

$$\begin{aligned} \int_{\mathbb{R}} Av \Delta^{-1/4} g dx &\lesssim \|A\|_{L^{2,1}} \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0, r))} \\ &+ \alpha^{1/2} \sum_{h=-1}^{+\infty} 2^{-h/2} \|A\|_{L^{2,1}} \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(A_{h,x_0})}. \end{aligned} \quad (83)$$

Proof of Lemma A.3. We write

$$\int_{\mathbb{R}} Av \Delta^{-1/4} g dx = \underbrace{\int_{\mathbb{R}} Av \mathbb{1}_{B(x_0, r)} \Delta^{-1/4} g dx}_{(9)} + \underbrace{\sum_{h=0}^{+\infty} \int_{\mathbb{R}} Av \mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g dx}_{(10)}$$

We have

$$\begin{aligned} (9) &\leq \|A \Delta^{-1/4} g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0, r))} \\ &\leq \|A\|_{L^{2,1}} \|\Delta^{-1/4} g\|_{L^\infty} \|v\|_{L^{2,\infty}(B(x_0, r))} \\ &\lesssim \|A\|_{L^{2,1}} \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0, r))}. \\ (10) &\simeq \sum_{h=0}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}[|\cdot|^{-1/2}](\xi) g * (\mathbb{1}_{A'_{h,x_0}} Av) d\xi \\ &\lesssim \sum_{h=0}^{+\infty} \|\xi|^{-1/2}\|_{L^\infty(B^c(0, 2^h r))} \|g * (\mathbb{1}_{A'_{h,x_0}} Av)\|_{L^1} \\ &\lesssim \sum_{h=0}^{+\infty} 2^{-h/2} r^{-1/2} \|g\|_{L^1} \|\mathbb{1}_{A'_{h,x_0}} Av\|_{L^1} \\ &\lesssim \sum_{h=0}^{+\infty} 2^{-h/2} r^{-1/2} (r\alpha)^{1/2} \|g\|_{L^{2,1}} \|A\|_{L^{2,1}} \|v\|_{L^{2,\infty}(A_{h',x_0})} \\ &\lesssim \alpha^{1/2} \|g\|_{L^{2,1}} \|A\|_{L^{2,1}} \sum_{h=0}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0})}. \end{aligned}$$

This concludes the proof of Lemma A.3. □

Lemma A.4 *Let $\Omega \in L^2(\mathbb{R}, \mathcal{M}_{m \times m}(\mathbb{R}))$ be such that $\text{supp } \Omega \subset B^c(0, \rho)$, $v \in L^2(\mathbb{R})$, $x_0 \in B(0, \rho/4)$, $g \in L^{2,1}(\mathbb{R})$, $\text{supp } g \subset B(x_0, r\alpha)$, with $0 < \alpha < 1$, $0 < r < \rho/8$.*

Then we have

$$\int_{\mathbb{R}} \Omega v \Delta^{-1/4} g dx \lesssim (r\alpha)^{1/2} \|g\|_{L^{2,1}} \|\Omega\|_{L^2} \|v\|_{L^2}. \quad (84)$$

Proof of Lemma A.4. We use the fact that Ω and g have disjoint supports.

$$\begin{aligned} \int_{\mathbb{R}} \Omega v \Delta^{-1/4} g dx &= \int_{\mathbb{R}} \mathcal{F}^{-1}(|\cdot|^{-1/2})(\xi) g * \Omega v d\xi \\ &\lesssim \| |x|^{-1/2} \|_{B^c(0, \rho/4)} \|g * \Omega v\|_{L^1} \\ &\lesssim \left(\frac{4}{\rho}\right)^{1/2} \|g\|_{L^1} \|\Omega v\|_{L^1} \\ &\lesssim (r\alpha)^{1/2} \|g\|_{L^{2,1}} \|\Omega\|_{L^2} \|v\|_{L^2}. \end{aligned}$$

This concludes the proof of Lemma A.4. \square

Now we are going to localize the operator F defined in (16).

Lemma A.5 *Let $Q \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\|Q\|_{L^2(\mathbb{R})} \leq \varepsilon_0$, $v \in L^2(\mathbb{R})$, $g \in L^{2,1}(\mathbb{R})$, $\text{supp } g \subset B(x_0, r\alpha)$, with $x_0 \in \mathbb{R}$, $0 < \alpha < \frac{1}{4}$, $r > 0$.*

Then we have

$$\begin{aligned} \int_{\mathbb{R}} F(Q, v) \Delta^{-1/4} g dx &\lesssim \varepsilon_0 \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B_r(x_0))} \\ &+ \alpha^{1/2} (\|Q\|_{L^2(\mathbb{R})} + \|Q\|_{L^\infty}) \|g\|_{L^{2,1}} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0})}. \end{aligned} \quad (85)$$

Proof of Lemma A.5. We take the scalar product of $F(Q, v)$ with $\Delta^{-1/4} g$ and we integrate. We get

$$\begin{aligned} \int_{\mathbb{R}} F(Q, v) \Delta^{-1/4} g dx &= \underbrace{\int_{\mathbb{R}} F(Q, \chi_r v) \Delta^{-1/4} g dx}_{(11)} \\ &+ \underbrace{\int_{\mathbb{R}} \sum_{k=1}^{+\infty} F(Q, \varphi_k v) \Delta^{-1/4} g dx}_{(12)}. \end{aligned}$$

To estimate (11) we use the fact that $F(Q, v) \in \dot{H}^{-1/2}(\mathbb{R})$ and

$$\|F(Q, v)\|_{\dot{H}^{1/2}(\mathbb{R})} \lesssim \|Q\|_{L^2(\mathbb{R})} \|v\|_{L^{2,\infty}}.$$

$$\begin{aligned} (11) &\leq \|\Delta^{-1/4} g\|_{\dot{H}^{1/2}(\mathbb{R})} \|Q\|_{L^2(\mathbb{R})} \|v\|_{L^{2,\infty}(B(x_0, r))} \\ &\lesssim \|g\|_{L^{2,1}} \|Q\|_{L^2(\mathbb{R})} \|v\|_{L^{2,\infty}(B(x_0, r))} \\ &\lesssim \varepsilon_0 \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0, r))}. \end{aligned}$$

Next we spilt (12) in two parts:

$$\begin{aligned}
(12) &= \underbrace{\sum_{k=1}^{\infty} \int_{\mathbb{R}} F(Q, \varphi_k v) \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g dx}_{(13)} \\
&+ \underbrace{\sum_{k=1}^{\infty} \sum_{h=-1}^{\infty} \int_{\mathbb{R}} F(Q, \varphi_k v) \mathbb{1}_{A'_{h, x_0}} \Delta^{-1/4} g dx}_{(14)}.
\end{aligned}$$

Estimate of (13):

$$\begin{aligned}
(13) &= \sum_{k=1}^{+\infty} \int_{\mathbb{R}} F(Q, \varphi_k v) \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g dx \\
&= \sum_{k=1}^{+\infty} \int_{\mathbb{R}} \mathcal{R}(Q) \mathcal{R}(\varphi_k v) \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g dx \\
&\simeq \sum_{k=1}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1} \left[\frac{\cdot}{|\cdot|} \right] (\xi) (\varphi_k v) * (Q \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g) d\xi \\
&\lesssim \sum_{k=1}^{+\infty} \left\| \frac{1}{\xi} \right\|_{L^\infty(B^c(x_0, 2^{k-1}r))} \|\varphi_k v\|_{L^1(\mathbb{R})} \|Q \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g\|_{L^1(\mathbb{R})} \\
&\lesssim \sum_{k=1}^{+\infty} 2^{-k} r^{-1} 2^{k/2} r^{1/2} r \alpha^{1/2} \|v\|_{L^{2,\infty}(A_{h, x_0})} \|Q\|_{L^\infty} \|g\|_{L^{2,1}} \\
&\lesssim (r\alpha)^{1/2} \|Q\|_{L^\infty} \|g\|_{L^{2,1}} \sum_{k=1}^{+\infty} 2^{-k/2} \|v\|_{L^{2,\infty}(A_{h, x_0})}.
\end{aligned}$$

The estimate of (14) is analogous of (4) in the proof of Lemma A.2 and we omit it. \square

Lemma A.6 *Let $Q \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\text{supp } Q \subset B^c(0, \rho)$ for some $\rho > 0$, $v \in L^2(\mathbb{R})$, $x_0 \in B(0, \rho/4)$, $g \in L^{2,1}(\mathbb{R})$, $\text{supp } g \subset B(x_0, r\alpha)$, with $0 < \alpha < 1$, $0 < r < \rho/8$.*

Then we have

$$\begin{aligned}
\int_{\mathbb{R}} F(Q, v) \Delta^{-1/4} g dx &\lesssim \left[\alpha^{1/2} + \left(\frac{r}{\rho} \right)^{1/2} \right] \|Q\|_{L^2} \|g\|_{L^{2,1}} \|v\|_{L^{2,\infty}(B(x_0, r))} \\
&+ \alpha^{1/2} (\|Q\|_{L^2} + \|Q\|_{L^\infty}) \|g\|_{L^{2,1}} \sum_{h=1}^{+\infty} 2^{-h/2} \|v\|_{L^{2,\infty}(A_{h,x_0})}.
\end{aligned} \tag{86}$$

Proof of Lemma A.6. We just give a sketch of proof.

We write

$$\begin{aligned}
\int_{\mathbb{R}} F(Q, v) \Delta^{-1/4} g dx &= \underbrace{\int_{\mathbb{R}} F(Q, \chi_r v) \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g dx}_{(15)} \\
&+ \underbrace{\int_{\mathbb{R}} F(Q, \chi_r v) \mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g dx}_{(16)} \\
&+ \underbrace{\int_{\mathbb{R}} F(Q, (1 - \chi_r) v) \Delta^{-1/4} g dx}_{(17)}.
\end{aligned}$$

To estimate (15) we write $Q = \sum_{h=-2} \tilde{\varphi}_h Q$ with $\text{supp } \tilde{\varphi}_h \subseteq B(0, 2^{h+1}\rho \setminus B(0, 2^{h-1}\rho))$ and $\tilde{\varphi}_h$ partition of unity.

$$\begin{aligned}
(15) &= \sum_{h=-2}^{\infty} \int_{\mathbb{R}} \mathcal{R}(\tilde{\varphi}_h Q) \mathcal{R}(\chi_r v) \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g dx \\
&= \sum_{h=-2}^{\infty} \int_{\mathbb{R}} \mathcal{F}^{-1} \left[\frac{\cdot}{|\cdot|} \right] (\xi) (\tilde{\varphi}_h Q) * [\mathcal{R}(\chi_r v) \mathbb{1}_{B(x_0, r/4)} \Delta^{-1/4} g] d\xi \\
&\lesssim \sum_{h=-2}^{\infty} \|\xi^{-1}\|_{L^\infty(B^c(0, 2^h \rho))} \|\tilde{\varphi}_h Q\|_{L^1} \|\mathcal{R}(\chi_r v)\|_{L^1(B(x_0, r/4))} \|\Delta^{-1/4} g\|_{L^\infty(\mathbb{R})} \\
&\lesssim \|g\|_{L^{2,1}} \|\mathcal{R}(\chi_r v)\|_{L^{2,\infty}(B(x_0, r/4))} \left(\frac{r}{\rho} \right)^{1/2} \sum_{h=-2}^{\infty} 2^{-h/2} \|Q\|_{L^2(A_{h,0})} \\
&\lesssim \left(\frac{r}{\rho} \right)^{1/2} \|g\|_{L^{2,1}} \|Q\|_{L^2} \|v\|_{L^{2,\infty}(B(x_0, r))}.
\end{aligned}$$

Now we write

$$\begin{aligned}
(16) &= \sum_{h=-2}^{+\infty} \sum_{k=-2}^{+\infty} \int_{\mathbb{R}} F(\tilde{\varphi}_k Q, \chi_r v) \mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g dx \\
&= \sum_{h=-2}^{+\infty} \sum_{|k-h| \leq 5} \int_{\mathbb{R}} F(\tilde{\varphi}_k Q, \chi_r v) \mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g dx \\
&\quad + \sum_{h=-2}^{+\infty} \sum_{|k-h| > 5} \int_{\mathbb{R}} F(\tilde{\varphi}_k Q, \chi_r v) \mathbb{1}_{A'_{h,x_0}} \Delta^{-1/4} g dx
\end{aligned}$$

by arguing as in (5) and (6)

$$\lesssim \|g\|_{L^{2,1}} \|Q\|_{L^2} \left[\left(\frac{r}{\rho} \right)^{1/2} + \alpha^{1/2} \right].$$

The estimate of (16) is analogous to (2) in the proof of Lemma A.1 and we omit it. \square

B Commutator Estimates

We consider the Littlewood-Paley decomposition of unity introduced in Section 2. For every $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the Littlewood-Paley projection operators P_j and $P_{\leq j}$ by

$$\widehat{P_j f} = \psi_j \hat{f} \quad \widehat{P_{\leq j} f} = \phi_j \hat{f}.$$

Informally P_j is a frequency projection to the annulus $\{2^{j-1} \leq |\xi| \leq 2^j\}$, while $P_{\leq j}$ is a frequency projection to the ball $\{|\xi| \leq 2^j\}$. We will set $f_j = P_j f$ and $f^j = P_{\leq j} f$.

We observe that $f^j = \sum_{k=-\infty}^j f_k$ and $f = \sum_{k=-\infty}^{+\infty} f_k$ (where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$).

Given $f, g \in \mathcal{S}'(\mathbb{R})$ we can split the product in the following way

$$fg = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g), \tag{87}$$

where

$$\begin{aligned}\Pi_1(f, g) &= \sum_{-\infty}^{+\infty} f_j \sum_{k \leq j-4} g_k = \sum_{-\infty}^{+\infty} f_j g^{j-4}; \\ \Pi_2(f, g) &= \sum_{-\infty}^{+\infty} f_j \sum_{k \geq j+4} g_k = \sum_{-\infty}^{+\infty} g_j f^{j-4}; \\ \Pi_3(f, g) &= \sum_{-\infty}^{+\infty} f_j \sum_{|k-j| < 4} g_k.\end{aligned}$$

We observe that for every j we have

$$\begin{aligned}\text{supp} \mathcal{F}[f^{j-4} g_j] &\subset \{2^{j-2} \leq |\xi| \leq 2^{j+2}\}; \\ \text{supp} \mathcal{F}[\sum_{k=j-3}^{j+3} f_j g_k] &\subset \{|\xi| \leq 2^{j+5}\}.\end{aligned}$$

The three pieces of the decomposition (87) are examples of paraproducts. Informally the first paraproduct Π_1 is an operator which allows high frequencies of f ($\sim 2^j$) multiplied by low frequencies of g ($\ll 2^j$) to produce high frequencies in the output. The second paraproduct Π_2 multiplies low frequencies of f with high frequencies of g to produce high frequencies in the output. The third paraproduct Π_3 multiply high frequencies of f with high frequencies of g to produce comparable or lower frequencies in the output. For a presentation of these paraproducts we refer to the reader for instance to the book [8]. The following two Lemmae will be often used in the sequel. For the proof of the first one we refer the reader to [4].

Lemma B.1 *For every $f \in \mathcal{S}'$ we have*

$$\sup_{j \in \mathbb{Z}} |f^j| \leq M(f).$$

Lemma B.2 *Let ψ be a Schwartz radial function such that $\text{supp}(\psi) \subset B(0, 4)$. Then*

$$\|\nabla^k \mathcal{F}^{-1} \psi\|_{L^1} \leq C_{\psi, n} 4^k,$$

where $C_{\psi, n}$ is a positive constant depending on the C^2 norm of ψ and the dimension.

Proof of Lemma B.2. We recall that

$$\nabla^k \mathcal{F}^{-1} \psi(\xi) = \mathcal{F}^{-1}[i^k x^k \psi](\xi).$$

We write

$$\int_{\mathbb{R}^n} |\nabla^k \mathcal{F}^{-1} \psi(\xi)| d\xi = \int_{|\xi| \leq 1} |\nabla^k \mathcal{F}^{-1} \psi(\xi)| d\xi + \int_{|\xi| \geq 1} |\nabla^k \mathcal{F}^{-1} \psi(\xi)| d\xi.$$

The following estimates hold.

$$\begin{aligned} \int_{|\xi| \leq 1} |\nabla^k \mathcal{F}^{-1} \psi(\xi)| d\xi &\leq \omega_n \|\nabla^k \mathcal{F}^{-1} \psi(\xi)\|_{L^\infty} \\ &\leq \omega_n \|x^k \psi\|_{L^1} \leq \omega_n 4^k \|\psi\|_{L^1}, \end{aligned} \quad (88)$$

where $\omega_n = |B_1(0)|$.

$$\begin{aligned} \int_{|\xi| \geq 1} |\nabla^k \mathcal{F}^{-1} \psi(\xi)| d\xi &= \int_{|\xi| \geq 1} \left(-\frac{1}{|\xi|^2}\right) \left[\int_{\mathbb{R}^n} (\Delta_x e^{i\xi x}) \psi(x) (ix)^k dx \right] d\xi \\ &= \int_{|\xi| \geq 1} \left(-\frac{1}{|\xi|^2}\right) \left[\int_{\mathbb{R}^n} e^{i\xi x} \Delta_\xi (\psi(x) (ix)^k) dx \right] d\xi \\ &\leq \int_{|\xi| \geq 1} \frac{1}{|\xi|^2} d\xi \left(\frac{4^k + 2k4^{k-1} + k(k-1)4^{k-2}}{|x|^2} \|\psi\|_{C^2} \right). \end{aligned} \quad (89)$$

By combining (88) and (89) we obtain

$$\begin{aligned} \|\nabla^k \mathcal{F}^{-1} \psi(\xi)\|_{L^1} &\leq \omega_n 4^k \|\psi\|_{L^1} \\ &\quad + \frac{4^k + 2k4^{k-1} + k(k-1)4^{k-2}}{|x|^2} \|\psi\|_{C^2} \int_{|\xi| \geq 1} \frac{1}{|\xi|^2} d\xi \\ &\leq C_{\psi,n} 4^k. \end{aligned} \quad (90)$$

This concludes the proof of Lemma B.2. \square

Lemma B.3 *Let $f \in B_{\infty,\infty}^0(\mathbb{R}^n)$. Then for all $k \in \mathbb{N}$ and for all $j \in \mathbb{Z}$ we have*

$$2^{-kj} \|\nabla^k f_j\|_{L^\infty} \leq 4^k \|f_j\|_{L^\infty}.$$

Proof of Lemma B.3. Let Ψ be a Schwartz radial function such that $\Psi = 1$ in B_2 and $\Psi = 0$ in $B^c(0,4)$.

Since $\text{supp } \mathcal{F}[f_j] \subseteq B_{2^{j+1}} \setminus B_{2^{j-1}}$ we have

$$\mathcal{F}[\nabla^k f_j] \simeq \xi^k \mathcal{F}[f_j] = 2^{kj} \psi(2^{-j}\xi) \frac{\xi^k}{2^{kj}} \mathcal{F}[f_j]. \quad (91)$$

Observe that

$$\begin{aligned} &\|\mathcal{F}^{-1}[\psi(2^{-j}\xi) \frac{\xi^k}{2^{kj}}]\|_{L^1} \\ &= \left\| \int_{\mathbb{R}^n} e^{ix\xi} \psi(2^{-j}\xi) \frac{\xi^k}{2^{kj}} d\xi \right\|_{L^1} \\ &= 2^{nj} \left\| \int_{\mathbb{R}^n} e^{i2^j x\xi} \psi(\xi) \xi^k d\xi \right\|_{L^1} \\ &= 2^{nj} \|\nabla^k \mathcal{F}^{-1}[\psi](2^j \cdot)\|_{L^1}. \end{aligned}$$

Thus

$$\begin{aligned}
2^{-kj} \|\nabla^k f_j\|_{L^\infty} &\leq \|\mathcal{F}^{-1}[\psi(2^{-j}\xi) \frac{\xi^k}{2^{kj}}] * f_j\|_{L^\infty} \\
&\leq \|\mathcal{F}^{-1}[\psi(2^{-j}\xi) \frac{\xi^k}{2^{kj}}]\|_{L^1} \|f_j\|_{L^\infty} \\
&= \|\mathcal{F}^{-1}[\nabla^k \psi]\|_{L^1} \|f_j\|_{L^\infty} \leq C_\psi 4^k \|f_j\|_{L^\infty} . \quad \square
\end{aligned}$$

Now we start with a series of preliminary Lemmae which will be crucial in the construction of the gauge P in Section 4

Lemma B.4 *Let $a \in \dot{W}^{1/2,r}(\mathbb{R})$, $r < 2$ and $b \in \dot{H}^{1/2}(\mathbb{R})$. Then*

$$\|\Delta^{1/4}(ab) - a\Delta^{1/4}b - (\Delta^{1/4}a)b\|_{L^r(\mathbb{R})} \leq C \|a\|_{\dot{W}^{1/2,r}(\mathbb{R})} \|b\|_{\dot{H}^{1/2}(\mathbb{R})} .$$

Proof of Lemma B.4.

• **Estimate of $\|\Pi_2(\Delta^{1/4}(ab))\|_{L^r}$.**

$$\begin{aligned}
\left\| \sum_j \Delta^{1/4}(a^{j-4}b_j) \right\|_{L^r}^r &\lesssim \int_{\mathbb{R}} \left(\sum_j 2^j |a^{j-4}|^2 |b_j|^2 \right)^{r/2} dx \\
&\lesssim \int_{\mathbb{R}} \sup_j |a^{j-4}|^r \left(\sum_j 2^j |b_j|^2 \right)^{r/2} dx
\end{aligned}$$

by Hölder Inequality

$$\begin{aligned}
&\lesssim \left(\int_{\mathbb{R}} \sup_j |a^{j-4}|^{\frac{2r}{2-r}} dx \right)^{\frac{2-r}{2}} \left(\int_{\mathbb{R}} \sum_j |b_j|^2 dx \right)^{r/2} \\
&\lesssim \|a\|_{L^{\frac{2r}{2-r}}}^r \|b\|_{\dot{H}^{1/2}}^r \lesssim \|a\|_{\dot{W}^{1/2,r}}^r \|b\|_{\dot{H}^{1/2}}^r .
\end{aligned}$$

In the last inequality we use the embedding $\dot{W}^{1/2,r}(\mathbb{R}) \hookrightarrow L^{\frac{2r}{2-r}}(\mathbb{R})$, (see for instance [2]).

• **Estimate of $\|\Pi_2(a\Delta^{1/4}b)\|_{L^r}$.**

$$\begin{aligned}
\|\Pi_2(a\Delta^{1/4}b)\|_{L^r}^r &\lesssim \int_{\mathbb{R}} \left(\sum_j |a^{j-4}|^2 |\Delta^{1/4}b_j|^2 dx \right)^{r/2} dx \\
&\lesssim \int_{\mathbb{R}} \sup_j |a^{j-4}|^2 \left(\sum_j |\Delta^{1/4}b_j|^2 \right)^{r/2} dx \\
&\lesssim \|a\|_{\dot{W}^{1/2,r}}^r \|b\|_{\dot{H}^{1/2}}^r .
\end{aligned}$$

• **Estimate of $\|\Pi_2((\Delta^{1/4}a)b)\|_{L^r}$.**

$$\begin{aligned}
\|\Pi_2((\Delta^{1/4}a)b)\|_{L^r}^r &\lesssim \int_{\mathbb{R}} \left(\sum_j |\Delta^{1/4}a^{j-4}|^2 |b_j|^2 \right)^{r/2} dx \\
&\lesssim \int_{\mathbb{R}} \sup_j (2^{-j/2} |\Delta^{1/4}a^{j-4}|)^r \left(\sum_j 2^j |b_j|^2 \right)^{r/2} dx \\
&\lesssim \left(\int_{\mathbb{R}} \sup_j (2^{-j/2} |\Delta^{1/4}a^{j-4}|)^{\frac{2r}{2-r}} \right)^{\frac{2-r}{2}} \left(\int_{\mathbb{R}} \sum_j 2^j |b_j|^2 dx \right)^{r/2} \\
&\lesssim \left[\int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4}a^{j-4}|^2 \right)^{\frac{r}{2-r}} dx \right]^{\frac{2-r}{r}} \|b\|_{\dot{H}^{1/2}}^r \\
&\lesssim \|a\|_{\dot{W}^{1/2,r}} \|b\|_{\dot{H}^{1/2}}^r.
\end{aligned}$$

• **Estimate of $\|\Pi_3(\Delta^{1/4}(ab))\|_{L^r}$.**

$$\begin{aligned}
\|\Pi_3(\Delta^{1/4}(ab))\|_{L^r}^r &\simeq \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sum_j (\Delta^{1/4}h) a_j b_j dx \\
&= \sup_{\|h\|_{L^{r'}} \leq 1} \left[\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} (\Delta^{1/4}h_k) a_j b_j dx + \int_{\mathbb{R}} \sum_j (\Delta^{1/4}h^{j-4}) a_j b_j dx \right]. \quad (92)
\end{aligned}$$

Now we estimate the last two terms in (92).

$$\begin{aligned}
\int_{\mathbb{R}} \sum_j \Delta^{1/4}h^{j-4} a_j b_j dx &\lesssim \int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4}h^{j-4}|^2 \right)^{1/2} \left(\sum_j 2^j |a_j|^2 |b_j|^2 \right)^{1/2} dx \\
&\lesssim \left[\int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4}h^{j-4}|^2 \right)^{r'/2} dx \right]^{1/r'} \left[\int_{\mathbb{R}} \left(\sum_j 2^j |a_j|^2 |b_j|^2 \right)^{r/2} dx \right]^{1/r} \\
&\lesssim \|h\|_{L^{r'}} \|b\|_{B_{\infty,\infty}^0} \|a\|_{\dot{W}^{1/2,r}}.
\end{aligned}$$

The estimate of $\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} (\Delta^{1/4}h_k) a_j b_j dx$ is similar and we omit it.

- **Estimate of $\|\Pi_3(a\Delta^{1/4}b)\|_{L^r}$.**

$$\begin{aligned}\|\Pi_3(a\Delta^{1/4}b)\|_{L^r}^r &\lesssim \int_{\mathbb{R}} \left| \sum_j a_j \Delta^{1/4} b_j \right|^r dx \\ &\lesssim \int_{\mathbb{R}} \left(\sum_j a_j^2 \right)^{r/2} \left(\sum_j |\Delta^{1/4} b_j|^2 \right)^{r/2} dx\end{aligned}$$

by Cauchy-Schwartz Inequality

$$\begin{aligned}&\lesssim \left(\int_{\mathbb{R}} \left(\sum_j a_j^2 \right)^{r/(2-r)} dx \right)^{(2-r)/2} \left(\int_{\mathbb{R}} \left(\sum_j |\Delta^{1/4} b_j|^2 \right)^{r/(2-r)} dx \right)^{(2-r)/2} \\ &\lesssim \|a\|_{L^{\frac{2r}{2-r}}}^r \|b\|_{\dot{H}^{1/2}}^r \lesssim \|a\|_{\dot{W}^{1/2,r}}^r \|b\|_{\dot{H}^{1/2}}^r.\end{aligned}$$

- **Estimate of $\|\Pi_3((\Delta^{1/4}a)b)\|_{L^r}$.**

$$\begin{aligned}\|\Pi_3((\Delta^{1/4}a)b)\|_{L^r}^r &\lesssim \int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4} a_j|^2 \right)^{r/2} \left(\sum_j 2^j |b_j|^2 \right)^{r/2} dx \\ &\lesssim \|a\|_{\dot{W}^{1/2,r}} \|b\|_{\dot{H}^{1/2}}^r.\end{aligned}$$

- **Estimate of $\|\Pi_1(a\Delta^{1/4}b)\|_{L^r}$.**

$$\begin{aligned}\|\Pi_1(a\Delta^{1/4}b)\|_{L^r} &\simeq \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sum_j \Delta^{1/4} b^{j-4} a_j h_j dx \\ &\lesssim \int_{\mathbb{R}} \sup_j |\Delta^{1/4} b^{j-4}| \left(\sum_j a_j^2 \right)^{1/2} \left(\sum_j h_j^2 \right)^{1/2} dx\end{aligned}$$

by generalized Hölder Inequality: $\frac{1}{2} + \frac{1}{r'} + \frac{2-r}{2r} = 1$

$$\lesssim \|b\|_{\dot{H}^{1/2}} \|a\|_{L^{\frac{2r}{2-r}}} \|h\|_{L^{r'}} \lesssim \|b\|_{\dot{H}^{1/2}} \|a\|_{\dot{W}^{1/2,r}}.$$

• **Estimate of $\|\Pi_1(\Delta^{1/4}(ab) - (\Delta^{1/4}a)b)\|_{L^r}$.**

$$\begin{aligned}
& \|\Pi_1(\Delta^{1/4}(ab) - (\Delta^{1/4}a)b)\|_{L^r} \\
&= \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} |h_k(\Delta^{1/4}(a_j b^{j-4}) - \Delta^{1/4}a_j b^{j-4})| dx \\
&= \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} b^{j-4}(\Delta^{1/4}(h_k a_j) - \Delta^{1/4}(a_j) h_k) d\xi \\
&\lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} \mathcal{F}[b^{j-4}] \mathcal{F}[\Delta^{1/4}h_k a_j - \Delta^{1/4}a_j h_k] d\xi \\
&= \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j| \leq 3} \mathcal{F}[b^{j-4}](\xi) \\
&\quad \left[\int_{\mathbb{R}^n} (\mathcal{F}[h_j](\eta) \mathcal{F}[a_j](\xi - \eta) (|\eta|^{1/2} - |\xi - \eta|^{1/2}) d\eta \right] d\xi.
\end{aligned} \tag{93}$$

Now we observe that in (93) we have $|\xi| \leq 2^{j-3}$ and $2^{j-2} \leq |\eta| \leq 2^{j+2}$. Thus $|\frac{\xi}{\eta}| \leq \frac{1}{2}$. Hence

$$\begin{aligned}
|\eta|^{1/2} - |\xi - \eta|^{1/2} &= |\eta|^{1/2} [1 - |1 - \frac{\xi}{\eta}|^{1/2}] \\
&= |\eta|^{1/2} \frac{\xi}{\eta} [1 + |1 - \frac{\xi}{\eta}|^{1/2}]^{-1} \\
&= |\eta|^{1/2} \sum_{k=0}^{\infty} \frac{c_k}{k!} (\frac{\xi}{\eta})^{k+1}.
\end{aligned} \tag{94}$$

We may suppose that $\sum_{k=0}^{\infty} \frac{c_k}{k!} (\frac{\xi}{\eta})^{k+1}$ is convergent if $|\frac{\xi}{\eta}| \leq \frac{1}{2}$, otherwise one may consider a different Littlewood-Paley decomposition by replacing the exponent $j - 4$ with $j - s$, $s > 0$ large enough. We introduce the following notation: for every $k \geq 0$ we set

$$S_k g = \mathcal{F}^{-1}[\xi^{-(k+1)} |\xi|^{1/2} \mathcal{F}g].$$

We note that if $h \in B_{\infty, \infty}^s$ then $S_k h \in B_{\infty, \infty}^{s+1/2+k}$ and if $h \in H^s$ then $S_k h \in H^{s+1/2+k}$. Moreover if $Q \in H^{1/2}$ then $\nabla^{k+1}(Q) \in H^{-k-1/2}$.

Next we continue with the proof of (93).

$$\begin{aligned} & \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} h_k (\Delta^{1/4} (a_j b^{j-4}) - (\Delta^{1/4} a_j) b^{j-4}) dx \\ & \lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} \int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} [\nabla^{\ell+1} b^{j-4} [(S_{\ell} h_k) a_j]](x) dx \end{aligned}$$

by Lemma B.3

$$\begin{aligned} & \lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} 2^{-4\ell} 4^{\ell+1} \|b\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} 2^{(\ell+1)j} |[(S_{\ell} h_k) a_j]](x)| dx \\ & \lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} 2^{-4\ell} 4^{\ell+1} \|b\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}} \sum_j |(2^{(1/2+\ell)j} S_{\ell} h_j)| |(2^{j/2} a_j)| dx \end{aligned} \quad (95)$$

by Schwartz Inequality

$$\lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} 2^{-4\ell} 4^{\ell+1} \|b\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}} \left(\sum_j 2^{2j(1/2+\ell)} |S_{\ell} h_j|^2 \right)^{1/2} \left(\sum_j 2^j a_j^2 \right)^{1/2} dx$$

by Hölder Inequality

$$\begin{aligned} & \lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} 2^{-4\ell} 4^{\ell+1} \|b\|_{B_{\infty,\infty}^0} \\ & \quad \left(\int_{\mathbb{R}} \left(\sum_j 2^{2j(1/2+\ell)} |S_{\ell} h_j|^2 \right)^{r'/2} \right)^{1/r'} \left(\int_{\mathbb{R}} \left(\sum_j 2^j a_j^2 \right)^{r/2} \right)^{1/r} \\ & \lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} 2^{-2\ell} \|b\|_{B_{\infty,\infty}^0} \|h\|_{L^{r'}} \|\Delta^{1/4} a\|_{L^r} \\ & \lesssim \|b\|_{\dot{H}^{1/2}} \|a\|_{\dot{W}^{1/2,r}} \end{aligned}$$

This concludes the proof of Lemma B.4. \square

Lemma B.5 *Let $1 < r < 2$, $a \in \dot{W}^{1/2,r}(\mathbb{R})$ and $b \in \dot{H}^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then*

$$\|ab\|_{\dot{W}^{1/2,r}} \leq C \|a\|_{\dot{W}^{1/2,r}} (\|b\|_{\dot{H}^{1/2}} + \|b\|_{L^\infty}).$$

Proof of Lemma B.5 . • Estimate of $\|\Pi_1(\Delta^{1/4}(ab))\|_{L^r}$.

$$\begin{aligned} \left\| \sum_j \Delta^{1/4}(a_j b^{j-4}) \right\|_{L^r}^r &\lesssim \int_{\mathbb{R}} \left(\sum_j |a_j|^2 |b^{j-4}|^2 \right)^{r/2} dx \\ &\quad \int_{\mathbb{R}} \sup_j |b^{j-4}|^r \left(\sum_j 2^j |a_j|^2 \right)^{r/2} dx \\ &\lesssim \int_{\mathbb{R}} |M(b)|^r \left(\sum_j 2^j |a_j|^2 \right)^{r/2} dx \leq \|b\|_{L^\infty}^r \int_{\mathbb{R}} \left(\sum_j 2^j |a_j|^2 \right)^{r/2} dx \\ &\lesssim \|b\|_{L^\infty}^r \|a\|_{\dot{W}^{1/2,r}}^r. \end{aligned}$$

• Estimate of $\|\Pi_2 \Delta^{1/4}(ab)\|_{L^r}$.

$$\begin{aligned} \left\| \sum_j \Delta^{1/4}(a^{j-4} b_j) \right\|_{L^r} &\simeq \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sum_j a^{j-4} b_j \Delta^{1/4} h_j \\ &\lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} \sup_j |a^{j-4}| \left(\sum_j 2^j |b_j|^2 \right)^{1/2} \left(\sum_j |h_j| \right)^{1/2} dx \\ &\lesssim \sup_{\|h\|_{L^{r'}} \leq 1} \int_{\mathbb{R}} |M(a)| \left(\sum_j 2^j |b_j|^2 \right)^{1/2} \left(\sum_j |h_j| \right)^{1/2} dx \end{aligned}$$

by generalized Hölder Inequality: $\frac{1}{r'} + \frac{1}{2} + \frac{2-r}{2r} = 1$

$$\lesssim \|b\|_{\dot{H}^{1/2}} \|a\|_{\dot{W}^{1/2,r}}.$$

• Estimate of $\|\Pi_3(\Delta^{1/4}(ab))\|_{L^r}$.

$$\begin{aligned}
& \left\| \sum_j \Delta^{1/4}(a_j b_j) \right\|_{L^r} \\
& \simeq \sup_{\|h\|_{L^{r'}} \leq 1} \left[\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} \Delta^{1/4}(a_j b_j) h_k dx + \int_{\mathbb{R}} \sum_j \Delta^{1/4}(a_j b_j) h^{j-4} dx \right] \\
& = \sup_{\|h\|_{L^{r'}} \leq 1} \left[\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} (a_j b_j) \Delta^{1/4} h_k dx + \int_{\mathbb{R}} \sum_j (a_j b_j) \Delta^{1/4} h^{j-4} dx \right]
\end{aligned}$$

We estimate the term $\int_{\mathbb{R}} \sum_j (a_j b_j) \Delta^{1/4} h^{j-4} dx$.

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \sum_j (a_j b_j) \Delta^{1/4} h^{j-4} dx \right| \\
& \lesssim \|b\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4} h^{j-4}|^2 \right)^{1/2} \left(\sum_j 2^j a_j^2 \right)^{1/2} dx \\
& \lesssim \|b\|_{B_{\infty,\infty}^0} \left(\int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4} h^{j-4}|^2 \right)^{r'/2} dx \right)^{1/r'} \left(\int_{\mathbb{R}} \left(\sum_j 2^j a_j^2 \right)^{r/2} dx \right)^{1/r} \\
& \lesssim \|b\|_{B_{\infty,\infty}^0} \|h\|_{L^{r'}} \|a\|_{\dot{W}^{1/2,r}}.
\end{aligned}$$

The term $\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} (a_j b_j) \Delta^{1/4} h_k dx$ is estimated in a similar way. Thus we get

$$\left\| \sum_j \Delta^{1/4}(a_j b_j) \right\|_{L^r} \lesssim \|b\|_{\dot{H}^{1/2}} \|a\|_{\dot{W}^{1/2,r}}.$$

This concludes the proof of Lemma B.5. □

Lemma B.6 *Let $1 < r < 2 < q$, $a \in \dot{W}^{1/2,r}(\mathbb{R})$ and $b \in \dot{W}^{1/2,q}(\mathbb{R})$ and $t = \frac{2rq}{2r+q(2-r)}$. Then*

$$\left\| \Delta^{1/4}(ab) - (\Delta^{1/4}a)b \right\|_{L^t(\mathbb{R})} \leq C \|a\|_{\dot{W}^{1/2,r}(\mathbb{R}^n)} \|b\|_{\dot{W}^{1/2,q}(\mathbb{R})}.$$

Proof of Lemma B.6.

• **Estimate of $\|\Pi_2(\Delta^{1/4}(ab))\|_{L^t}$.**

$$\begin{aligned}
& \left\| \sum_j \Delta^{1/4} (a^{j-4} b_j) \right\|_{L^t}^t \lesssim \int_{\mathbb{R}} \left(\sum_j 2^j |a^{j-4}|^2 |b_j|^2 \right)^{t/2} dx \\
& \lesssim \int_{\mathbb{R}} \sup_j |a^{j-4}|^t \left(\sum_j 2^j |b_j|^2 \right)^{t/2} dx \\
& \lesssim \left(\int_{\mathbb{R}} M(a)^{\frac{tq}{q-t}} dx \right)^{1-\frac{t}{q}} \left(\int_{\mathbb{R}} \left(\sum_j 2^j |b_j|^2 \right)^{q/2} dx \right)^{t/q} \\
& \lesssim \|a\|_{L^{\frac{2r}{2-r}}}^t \|b\|_{\dot{W}^{1/2,q}}^t.
\end{aligned}$$

In the above expression we use the fact that $\frac{tq}{q-t} = \frac{2r}{2-r}$.

• **Estimate of $\|\Pi_2((\Delta^{1/4}a)b)\|_{L^t}$.**

$$\begin{aligned}
& \left\| \sum_j (\Delta^{1/4} a^{j-4}) b_j \right\|_{L^t}^t \\
& \lesssim \int_{\mathbb{R}} \left(\sup_j 2^{-j/2} |\Delta^{1/4} a^{j-4}| \right)^t \left(\sum_j 2^j |b_j|^2 \right)^{t/2} dx \\
& \lesssim \int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4} a^{j-4}|^2 \right)^{t/2} \left(\sum_j 2^j |b_j|^2 \right)^{t/2} dx \\
& \lesssim \left(\int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4} a^{j-4}|^2 \right)^{tq/2(q-t)} dx \right)^{1-t/q} \left(\int_{\mathbb{R}} \left(\sum_j 2^j |b_j|^2 \right)^{q/2} dx \right)^{t/q} \\
& \lesssim \|a\|_{L^{tq/q-t}}^t \|b\|_{\dot{W}^{1/2,q}}^t \lesssim \|a\|_{\dot{W}^{1/2,r}}^t \|b\|_{\dot{W}^{1/2,q}}^t.
\end{aligned}$$

• **Estimate of $\|\Pi_3(\Delta^{1/4}(ab))\|_{L^t}$.**

$$\begin{aligned}
& \left\| \sum_j \Delta^{1/4}(a_j b_j) \right\|_{L^t} \simeq \sup_{\|h\|_{L^{t'}} \leq 1} \int_{\mathbb{R}} \Delta^{1/4} h \sum_j a_j b_j dx \\
& \lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \left[\int_{\mathbb{R}} \sum_j \sum_{|j-k| \leq 4} \Delta^{1/4} h_k a_j b_j dx + \int_{\mathbb{R}} \sum_j \Delta^{1/4} h^{j-4} a_j b_j dx \right].
\end{aligned}$$

We estimate the term $\int_{\mathbb{R}} \sum_j \Delta^{1/4} h^{j-4} a_j b_j dx$.

$$\begin{aligned}
& \int_{\mathbb{R}} \sum_j \Delta^{1/4} h^{j-4} a_j b_j dx \\
& \lesssim \int_{\mathbb{R}} \sup_j (2^{-j/2} |\Delta^{1/4} h^{j-4}|) \left| \sum_j 2^{j/2} |a_j| |b_j| \right| dx \\
& \lesssim \int_{\mathbb{R}} \left(\sum_j |\Delta^{1/4} h^{j-4}|^2 \right)^{1/2} \left(\sum_j |a_j|^2 \right)^{1/2} \left(\sum_j 2^j |b_j|^2 \right)^{1/2} dx \\
& \lesssim \left[\int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4} h^{j-4}|^2 \right)^{t'/2} dx \right]^{1/t'} \left[\int_{\mathbb{R}} \left(\sum_j |a_j|^2 \right)^{t/2} \left(\sum_j 2^j |b_j|^2 \right)^{t/2} dx \right]^{1/t} \\
& \lesssim \|h\|_{L^{t'} \varepsilon_1} \left[\int_{\mathbb{R}} \left(\sum_j |a_j|^2 \right)^{tq/2(q-t)} dx \right]^{\frac{q-t}{qt}} \left[\int_{\mathbb{R}} \left(\sum_j 2^j |b_j|^2 \right)^{q/2} dx \right]^{1/q} \\
& \lesssim \|a\|_{tq/q-t} \|b\|_{W^{1/2,q}} \\
& \lesssim \|a\|_{W^{1/2,r}} \|b\|_{W^{1/2,q}}.
\end{aligned}$$

The estimate of $\int_{\mathbb{R}} \sum_j \sum_{|j-k| \leq 4} \Delta^{1/4} h_k a_j b_j dx$ is similar.

- **Estimate of $\|\Pi_3((\Delta^{1/4} a)b)\|_{L^t}$.**

$$\begin{aligned}
& \left\| \sum_j (\Delta^{1/4} a_j) b_j \right\|_{L^t}^t \lesssim \int_{\mathbb{R}} \left| \sum_j \Delta^{1/4} a_j b_j \right|^t \\
& \lesssim \int_{\mathbb{R}} \left(\sum_j 2^{-j} |\Delta^{1/4} a_j|^2 \right)^{t/2} \left(\sum_j 2^j |b_j|^2 \right)^{t/2} dx \\
& \lesssim \|a\|_{tq/q-t}^t \|b\|_{W^{1/2,q}}^t \lesssim \|a\|_{W^{1/2,r}}^t \|b\|_{W^{1/2,q}}^t.
\end{aligned}$$

• **Estimate of $\|\Pi_2(\Delta^{1/4}(ab) - (\Delta^{1/4}a)b)\|_{L^t}$.**

$$\begin{aligned}
& \left\| \sum_j (\Delta^{1/4}(ab) - (\Delta^{1/4}a)b) \right\|_{L^t} \\
& = \sup_{\|h\|_{L^{t'}} \leq 1} \int_{\mathbb{R}} \sum_j h_j [\Delta^{1/4}(a_j b^{j-4}) - (\Delta^{1/4}a_j) b^{j4}] dx \\
& = \sup_{\|h\|_{L^{t'}} \leq 1} \int_{\mathbb{R}} \sum_j b^{j-4} [(\Delta^{1/4}h_j)a_j - h_j(\Delta^{1/4}a_j)] dx \\
& = \sup_{\|h\|_{L^{t'}} \leq 1} \int_{\mathbb{R}} \sum_j \mathcal{F}[b]^{j-4}(\eta) \left(\int_{\mathbb{R}} \mathcal{F}[h]_j(\xi) \mathcal{F}[a]_j(\eta - \xi) [|\xi|^{1/2} - |\eta - \xi|^{1/2}] d\xi \right) d\eta.
\end{aligned} \tag{96}$$

Now we argue as in (95)

$$\begin{aligned}
(96) & \lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} \int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 3} [\nabla^{\ell+1} b^{j-4} [(S_{\ell} h_k) a_j]](x) dx \\
& \lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} \int_{\mathbb{R}} \sum_j [2^{-(\ell+1/2)j} \nabla^{\ell+1} b^{j-4} [2^{j(\ell+1/2)} (S_{\ell} h_j) a_j]](x) dx \\
& \lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} \int_{\mathbb{R}} \sup_j [2^{j(\ell+1/2)} (S_{\ell} h_j)] \\
& \quad \left(\sum_j |a_j|^2 \right)^{1/2} \left(\sum_j 2^{-2(\ell+1/2)j} |\nabla^{\ell+1} b^{j-4}|^2 \right)^{1/2} dx
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \sum_{\ell=0}^{\infty} \frac{C_{\ell}}{\ell!} 2^{-2\ell} \int_{\mathbb{R}} \left(\sum_j 2^{-2(\ell+1/2)j} |S_{\ell} h_j|^2 \right)^{1/2} \\
&\quad \left(\sum_j |a_j|^2 \right)^{1/2} \left(\sum_j 2^{-2(\ell+1/2)j} |\nabla^{\ell+1} b^{j-4}|^2 \right)^{1/2} dx \\
&\lesssim \sup_{\|h\|_{L^{t'}} \leq 1} \left(\int_{\mathbb{R}} \left(\sum_j 2^j |\Delta^{-1/4} h_j|^2 \right)^{t'/2} \right)^{t'} \\
&\quad \left[\int_{\mathbb{R}} \left(\sum_j |a_j|^2 \right)^{qt/2(q-t)} \right]^{q-t/qt} \left[\int_{\mathbb{R}} \left(\sum_j |b_j|^2 \right)^{q/2} \right]^{1/q} \\
&\lesssim \|a\|_{L^{qt/q-t}} \|b\|_{W^{1/2,q}} \lesssim \|a\|_{W^{1/2,r}} \|b\|_{W^{1/2,q}}.
\end{aligned}$$

This concludes the proof of Lemma B.6. \square

Lemma B.7 *Let $a \in \dot{H}^{1/2}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$, $b \in W^{1/2,q}(\mathbb{R}^n)$, $2 < q < +\infty$. Then*

$$\|\Delta^{1/4}(ab) - (\Delta^{1/4}a)b\|_{L^q(\mathbb{R}^n)} \leq \|b\|_{\dot{W}^{1/2,q}(\mathbb{R}^n)} \left[\|a\|_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|a\|_{L^{\infty}(\mathbb{R}^n)} \right].$$

Proof of Lemma B.7.

• **Estimate of $\|\Pi_1(\Delta^{1/4}(ab))\|_{L^q}^q$.**

$$\begin{aligned}
\left\| \sum_j \Delta^{1/4}(a^{j-4}b_j) \right\|_{L^q}^q &\lesssim \int_{\mathbb{R}} \left(\sum_j 2^j |a^{j-4}|^2 |b_j|^2 \right)^{q/2} \\
&\lesssim \|a\|_{L^{\infty}}^q \|b\|_{W^{1/2,q}}^q.
\end{aligned}$$

• **Estimate of $\|\Pi_1((\Delta^{1/4}a)b)\|_{L^q}^q$.**

$$\begin{aligned}
\left\| \sum_j \Delta^{1/4} a^{j-4} b_j \right\|_{L^q}^q &\lesssim \int_{\mathbb{R}} \left(\sum_j |\Delta^{1/4} a^{j-4}|^2 |b_j|^2 \right)^{q/2} \\
&\lesssim \left\| \sup_j 2^{-j/2} |\Delta^{1/4} a^{j-4}| \right\|_{L^{\infty}}^q \int_{\mathbb{R}} \left(\sum_j 2^j |b_j|^2 \right)^{q/2} dx \\
&\lesssim \|b\|_{W^{1/2,q}}^q \|a\|_{B_{\infty,\infty}^0}.
\end{aligned}$$

• **Estimate of $\|\Pi_3(\Delta^{1/4}(ab))\|_{L^q}$.**

$$\begin{aligned} \left\| \sum_j \Delta^{1/4}(a_j b_j) \right\|_{L^q} &= \sup_{\|h\|_{L^{q'}} \leq 1} \int_{\mathbb{R}} (\Delta^{1/4} h) \sum_j a_j b_j dx \\ &= \sup_{\|h\|_{L^{q'}} \leq 1} \left[\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 4} (\Delta^{1/4} h_k) a_j b_j dx + \int_{\mathbb{R}} \sum_j (\Delta^{1/4} h^{j-4}) a_j b_j dx \right]. \end{aligned}$$

We estimate the last term:

$$\begin{aligned} &\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 4} \Delta^{1/4} h^{j-4} a_j b_j dx \\ &\lesssim \|a\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}} \left| \sum_j 2^{-j} |\Delta^{1/4} h^{j-4}|^2 \right|^{1/2} \int_{\mathbb{R}} \left| \sum_j 2^j |b_j|^2 \right|^{1/2} \\ &\lesssim \|a\|_{B_{\infty,\infty}^0} \left(\int_{\mathbb{R}} \left| \sum_j 2^{-j} |\Delta^{1/4} h^{j-4}|^2 \right|^{q'/2} \right)^{1/q'} \left(\int_{\mathbb{R}} \left| \sum_j 2^j |b_j|^2 \right|^{q/2} \right)^{1/q} \\ &\lesssim \|a\|_{B_{\infty,\infty}^0} \|b\|_{W^{1/2,q}}. \end{aligned}$$

• **Estimate of $\|\Pi_3((\Delta^{1/4}a)b)\|_{L^q}$.**

$$\begin{aligned} \left\| \sum_j \Delta^{1/4} a_j b_j \right\|_{L^q} &= \sup_{\|h\|_{L^{q'}} \leq 1} \int_{\mathbb{R}} h \sum_j \Delta^{1/4} a_j b_j dx \\ &= \sup_{\|h\|_{L^{q'}} \leq 1} \left[\int_{\mathbb{R}} \sum_j \sum_{|k-j| \leq 4} h_k (\Delta^{1/4} a_j) b_j dx + \int_{\mathbb{R}} \sum_j h^{j-4} (\Delta^{1/4} a_j) b_j dx \right] \end{aligned}$$

We estimate the last term $\int_{\mathbb{R}} \sum_j h^{j-4} \Delta^{1/4} a_j b_j dx$.

To this purpose we show that $\sum_j \Delta^{1/4}(h^{j-4} b_j) \in h^1$ and the conclusion follows from the

embedding $\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R})$. We have

$$\begin{aligned}
& \int_{\mathbb{R}} \left(\sum_j 2^j |h^{j-4} b_j|^2 \right)^{1/2} dx \\
& \lesssim \int_{\mathbb{R}} \sup_j |h^{j-4}| \left(\sum_j 2^j |b_j|^2 \right)^{1/2} dx \\
& \lesssim \left(\int_{\mathbb{R}} \sup_j |h^{j-4}|^{q'} \right)^{1/q'} \left(\int_{\mathbb{R}} \left(\sum_j 2^j |b_j|^2 \right)^{q/2} \right)^{1/q} \\
& \lesssim \|h\|_{L^{q'}} \|b\|_{W^{1/2,q}}.
\end{aligned}$$

• **Estimate of $\|\Pi_2(\Delta^{1/4}(ab) - (\Delta^{1/4}a)b)\|_{L^q}$.**

$$\begin{aligned}
& \|\Pi_2(\Delta^{1/4}(ab) - (\Delta^{1/4}a)b)\|_{L^q} \\
& \simeq \sup_{\|h\|_{L^{q'}} \leq 1} \int_{\mathbb{R}} \sum_j h_j (\Delta^{1/4}(a_j b^{j-4}) - \Delta a_j b^{j-4}) dx \\
& \simeq \sup_{\|h\|_{L^{q'}} \leq 1} \int_{\mathbb{R}} \sum_j b^{j-4} (\Delta^{1/4}(h_j) a_j - h_j \Delta a_j) dx \\
& \sup_{\|h\|_{L^{q'}} \leq 1} \int_{\mathbb{R}} \sum_j \mathcal{F}[b]^{j-4}(\eta) \int_{\mathbb{R}} \mathcal{F}[h]_j(\xi) \mathcal{F}[a]_j(\eta - \xi) (|\xi|^{1/2} - |\eta - \xi|^{1/2}) d\xi \\
& \dots \\
& \lesssim \|a\|_{B_{\infty,\infty}^0} \|b\|_{W^{1/2,q}} \|h\|_{L^{q'}}.
\end{aligned}$$

This concludes the proof of Lemma B.7. \square

In the next Theorem we prove an estimate for the dual of the operator F introduced in (16). It is defined as follows: given $Q \in L^2(\mathbb{R})$, $v \in \dot{H}^{1/2}(\mathbb{R})$ we have

$$F^*(Q, v) = \Delta^{1/4}(Qv) - \Delta^{1/4}\mathcal{R}(\mathcal{R}(Q)v).$$

Lemma B.8 *Let $Q \in L^2(\mathbb{R})$, $v \in \dot{H}^{1/2}(\mathbb{R})$. Then*

$$\|\Delta^{1/4}(Qv) - \Delta^{1/4}\mathcal{R}(\mathcal{R}(Q)v)\|_{\mathcal{H}^1} \lesssim \|Q\|_{L^2} \|v\|_{\dot{H}^{1/2}}. \quad (97)$$

Proof of Lemma B.8 .

Estimate of $\Pi_2(\Delta^{1/4}(Q, v))$.

$$\begin{aligned}
\|\Pi_1(\Delta^{1/4}(Q, v))\|_{\mathcal{H}^1} &= \int_{\mathbb{R}} \left(\sum_{i=-\infty}^{+\infty} 2^i (Q^{i-4})^2 (v_i)^2 \right)^{1/2} dx \\
&\lesssim \int_{\mathbb{R}} |M(Q)| \left(\sum_{i=-\infty}^{+\infty} 2^i (v_i)^2 \right) dx \\
&\lesssim \|Q\|_{L^2} \|v\|_{\dot{H}^{1/2}} .
\end{aligned} \tag{98}$$

The estimate of $\Pi_1(\Delta^{1/4}\mathcal{R}(\mathcal{R}(Q)v))$ is analogous to (98) .

Estimate of $\Pi_3(\Delta^{1/4}(Q, v))$.

$$\begin{aligned}
\|\Pi_3(Q, v)\|_{B_{1,1}^0} &\simeq \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \int_{\mathbb{R}} (Q_i v_i) \left[\Delta^{1/4} h^{i-6} + \sum_{t=h-5}^{i+6} \Delta^{1/4} h_t \right] dx \\
&\lesssim \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \|h\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}} 2^{i/2} |Q_i v_i| dx \\
&\lesssim \left(\int_{\mathbb{R}} \sum_i 2^i v_i^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \sum_i Q_i^2 dx \right)^{1/2} = \|Q\|_{L^2} \|v\|_{\dot{H}^{1/2}} .
\end{aligned} \tag{99}$$

The estimate of $\Pi_3(\Delta^{1/4}\mathcal{R}(\mathcal{R}(Q)v))$ is analogous to (99) .

Finally one can easily check that

$$\|\Pi_1(\Delta^{1/4}(Qv) - \Delta^{1/4}\mathcal{R}(\mathcal{R}(Q)v))\|_{\mathcal{H}^1} = 0 .$$

This concludes the proof of Lemma B.8 . □

References

- [1] D.R. Adams *A note on Riesz potentials*. Duke Math. J. 42 (1975), no. 4, 765–778.
- [2] D.R. Adams & L. I. Hedberg *Function Spaces and Potential Theory*, 1996, Springer, Berlin .
- [3] R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for hardy spaces in several variables*. Ann. of Math. 103 (1976), 611-635.
- [4] F. Da Lio & T. Riviere *3-Commutators Estimates and the Regularity of 1/2-Harmonic Maps into Spheres*.to appear in Analysis and PDE .

- [5] F. Da Lio & T. Riviere In preparation
- [6] H. Federer *Geometric Measure Theory*. Springer 1969. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
- [7] L. Grafakos *Classical Fourier Analysis*. Graduate Texts in Mathematics 249, Springer (2009).
- [8] L. Grafakos *Modern Fourier Analysis*. Graduate Texts in Mathematics 250, Springer (2009).
- [9] R. Moser *Intrinsic semiharmonic maps* preprint, 2009 (to appear in J. Geom. Anal.)
- [10] T. Rivière, *Conservation laws for conformal invariant variational problems* , Invent. Math., 168 (2007), 1-22.
- [11] T. Rivière, *Sub-criticality of Schrödinger Systems with Antisymmetric Potentials*, to appear in J. Math. Pures Appl
- [12] T. Rivière, *The role of Integrability by Compensation in Conformal Geometric Analysis*, to appear in Analytic aspects of problems from Riemannian Geometry S.M.F. (2008)
- [13] T. Rivière, M. Struwe, *Partial regularity for harmonic maps, and related problems*. Comm. Pure and Applied Math., 61 (2008), no. 4, 451-463
- [14] T. Runst & W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*. Walter de Gruyter, Berlin , 1996
- [15] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*. Lecture Notes of the Unione Matematica Italiana, 3. Springer, Berlin; UMI, Bologna, 2007.
- [16] K. Uhlenbeck *Connections with L^p bounds on curvature* Comm. Math. Phys, 83, 31-42, 1982.